

Game-theoretic approaches to partial differential equations and related problems

Jeongmin Han

Department of Mathematics,
Soongsil University G-LAMP Group

Recent Research in AI: Mathematics, Statistics and Chemistry
New Uzbekistan University (NUU), Tashkent

October 3, 2026

Mean value property

Every harmonic function, i.e. a solution of the Laplace equation

$$\Delta u = 0$$

satisfies the following mean value property.

Theorem (Mean value property)

Let Ω be a domain, $r > 0$ and u be a harmonic function. Then, for each $x \in \Omega$ with $B_r(x) \subset\subset \Omega$, we have

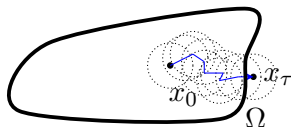
$$u(x) = \int_{B_r(x)} u(y) dy.$$

Mean value property

Let u be a harmonic function in a bounded domain Ω . For each $x_0 \in \Omega$, assume that X is randomly chosen according to the uniform distribution in $B_r(x_0)$. Then, we see that

$$\mathbb{E}^{x_0}[u(X)] = \int_{B_r(x_0)} u(y) dy = u(x_0).$$

Based on the above observation, we can construct a stochastic process for the Laplacian with proper boundary data.



$$u_\varepsilon(x_0) = \mathbb{E}^{x_0}[u_\varepsilon(x_\tau)]!$$

The p -Laplacian

The p -Laplacian is defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du).$$

If u is a C^2 -function with $Du \neq 0$, we can also write

$$\begin{aligned}\Delta_p u &= |Du|^{p-2} (\Delta u + (p-2) \Delta_\infty^N u) \\ &= |Du|^{p-2} \left(\Delta u + (p-2) \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} \right) \\ &=: |Du|^{p-2} \Delta_p^N u.\end{aligned}$$

We call Δ_p^N the normalized p -Laplacian.

Mean value characterizations for p -harmonic functions

Let $\Omega \subset \mathbb{R}^n$ and u be a solution of

$$\Delta_p^N u = 0 \quad \text{in } \Omega$$

for $2 \leq p < \infty$.

Then, for small $r > 0$, one can observe the following mean value characterization (by Manfredi-Parviainen-Rossi)

$$u(x) = \frac{\alpha}{2} \left(\sup_{B_r(x)} u + \inf_{B_r(x)} u \right) + \beta \int_{B_r(x)} u(y) dy + o(r^2), \quad (2.1)$$

with $\alpha = \frac{p-2}{p+n}$ and $\beta = \frac{n+2}{p+n}$ for any x with $B_r(x) \subset\subset \Omega$.

Example: time-independent tug-of-war with noise

It is natural to consider the following dynamic programming principle (DPP)

$$u_\varepsilon(x) = \frac{\alpha}{2} \left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon \right) + \beta \int_{B_\varepsilon(x)} u_\varepsilon(y) dy \quad (2.2)$$

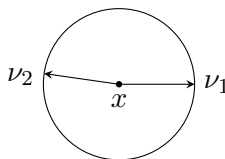
for small $\varepsilon > 0$.

- Q) How can we construct a stochastic process associated with (2.2)?

Example: time-independent tug-of-war with noise

Peres, Schramm, Sheffield and Wilson [PSSW09] presented stochastic interpretations of Δ_∞^N by using 'tug-of-war game' with the following DPP

$$u_\varepsilon(x) = \frac{1}{2} \left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon \right).$$

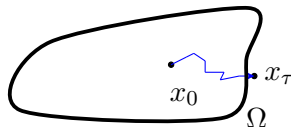


Example: time-independent tug-of-war with noise

A stochastic game related (2.2) (cf. Peres-Sheffield [PS08]):

- ▶ 'Tug-of-war' with a probability α .
- ▶ 'Random walk' with a probability β .

$$u_\varepsilon(x_0) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_0} [u_\varepsilon(x_\tau)].$$



Example: time-independent tug-of-war with noise

There are several issues related to game value functions.

- ▶ How can we deal with game values?
 - The DPP corresponding to the mean value characterization of the model equation
- ▶ Which properties do value functions have?
 - Existence and uniqueness, regularity, long-time behaviors (for time-dependent games)...
- ▶ How is a stochastic game related to its model equation?
 - Convergence of the value function to the solution of the model equation when ε is shrinking

Example: time-independent tug-of-war with noise

For existence:

- Constructing a supermartingale and applying the optional stopping theorem

For uniqueness:

- Using the maximum principle of the value function
(For the time-dependent case, it is enough to use mathematical induction)

Example: time-independent tug-of-war with noise

For regularity:

► Interior case

- Set $g(x, z) = u_\varepsilon(x) - u_\varepsilon(z)$ and consider the $2n$ -dimensional DPP which g satisfies
- Construct a proper auxiliary function f
- Estimate $\sup(g - f)$ by ε -terms

► Boundary case

- Estimate the stopping time of the game with a starting point near the boundary
(Regularity conditions for the domain and the boundary data are needed)

Example: time-independent tug-of-war with noise

For convergence of the value function:

- ▶ We can construct a convergent sequence of value functions by using an Arzelà-Ascoli criterion provided proper regularity results for the value function
- ▶ The notion of viscosity solution is necessary to verify that the limit function solves the model problem

Time-independent tug-of-war games

We also studied a tug-of-war game related to the problem

$$\begin{cases} \Delta_p^N u = 0 & \text{in } \Omega, \\ \langle \beta, Du \rangle + \gamma u = F & \text{on } \partial\Omega \end{cases} \quad (3.3)$$

for $2 < p < \infty$ in [Han25b].

For the game value functions related to (3.3), we obtained

- ▶ The existence and uniqueness
- ▶ The interior and boundary regularity estimates
- ▶ Convergence to the solution of the model equation

Time-independent tug-of-war games

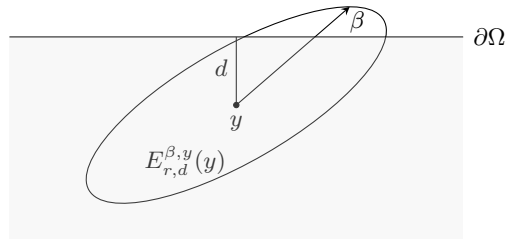
For a game related to (3.3), we considered the following DPP

$$\begin{aligned} u(x) = & (1 - \gamma s_\varepsilon(x)) \\ & \times \left\{ \frac{\alpha}{2} \left(\sup_{B_{\varepsilon d'_x}(x)} u + \inf_{B_{\varepsilon d'_x}(x)} u \right) + (1 - \alpha) \int_{E_\varepsilon^{\beta, x}(x) \cap \Omega} u(y) dy \right\} \\ & + \gamma s_\varepsilon(x) G(x), \end{aligned}$$

where $d'_\varepsilon(x) = \min \left\{ \frac{1}{2}, \frac{1}{\varepsilon} \text{dist}(x, \partial\Omega) \right\}$ and

$$s_\varepsilon(x) = \frac{|B_1^{n-1}|}{(n+1)|B_{1, d_\varepsilon(x)}^n|} \varepsilon (1 - d_\varepsilon(x)^2)^{\frac{n+1}{2}}.$$

Time-independent tug-of-war games



Time-independent tug-of-war games

For the boundary-value problem

$$\begin{cases} \Delta_p^N u - cu = 0 & \text{in } \Omega, \\ u = F & \text{on } \partial\Omega \end{cases} \quad (3.4)$$

with $2 < p < \infty$, we also investigated its game-theoretic approach in [Han25a].

For the game value functions related to (3.4), we obtained

- ▶ The existence and uniqueness
- ▶ The interior and boundary regularity estimates
- ▶ Convergence to the solution of the model equation

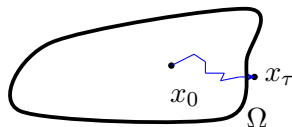
Time-independent tug-of-war games

We considered the DPP

$$u_\varepsilon(x) = (1 - \gamma\varepsilon^2) \left\{ \frac{\alpha}{2} \left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon \right) + \beta \int_{B_\varepsilon(x)} u_\varepsilon(y) dy \right\}.$$

Under the game setting, the value function is defined by

$$u_\varepsilon(x_0) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}_{S_I, S_{II}}^{x_0} [(1 - \gamma\varepsilon^2)^\tau u_\varepsilon(x_\tau)].$$



Tug-of-war on graphs

In [HL25], we studied the consistency of the following tug-of-war operators

$$\mathcal{L}_{\mathcal{X}_n, \epsilon} u(x) = \frac{\alpha}{2\epsilon^2} \left\{ \inf_{Z_i \in B_\epsilon^{\mathcal{X}_n}(x)} u(Z_i) + \sup_{Z_j \in B_\epsilon^{\mathcal{X}_n}(x)} u(Z_j) - 2u(x) \right\} \\ + \frac{\beta}{\text{card}(B_\epsilon^{\mathcal{X}_n}(x))} \sum_{Z_i \in B_\epsilon^{\mathcal{X}_n}(x)} \frac{u(Z_i) - u(x)}{\epsilon^2}$$

associated with the weighted p -Laplace operator

$$\phi^{-2} |Du|^{2-p} \operatorname{div}(\phi^2 |Du|^{p-2} Du)$$

(cf. [ABP24]).

Time-dependent tug-of-war games

We have considered a tug-of-war game associated with the following problem

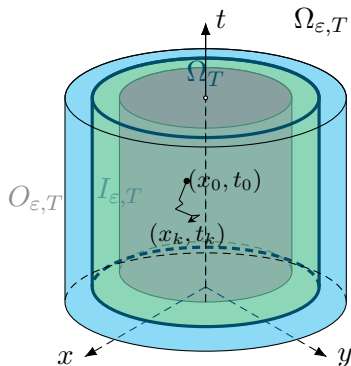
$$\begin{cases} u_t = \Delta_p^N u & \text{in } \Omega_T = \Omega \times (0, T), \\ u = F & \text{on } \partial_p \Omega_T \end{cases} \quad (3.5)$$

for $1 < p < \infty$ in [Han20, Han22].

For the game value functions related to (3.5), we obtained

- ▶ The existence and uniqueness
- ▶ The interior and boundary regularity estimates
- ▶ Long-time behavior
- ▶ Convergence to the solution of the model equation

Time-dependent tug-of-war games



Eigenvalues of Hessian matrices

For an $n \times n$ -matrix M , let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of M with $\lambda_1 \leq \dots \leq \lambda_n$.

One can observe that

$$\lambda_j(M) = \inf_{\dim(S)=j} \sup_{|v|=1} \langle Mv, v \rangle$$

and

$$\frac{u(x + \varepsilon v) + u(x - \varepsilon v)}{2} \approx \varepsilon^2 \langle D^2 u(x) v, v \rangle.$$

- Blanc and Rossi [BR19] studied game-theoretic approaches for the PDE $\lambda_j(D^2 u) = 0$, where $j = 1, \dots, n$ (cf. [BCMR23]).

Games for convex combinations of eigenvalues

We give a game-theoretic approach for the following fully nonlinear PDE

$$\sum_{i=1}^n \alpha_i \lambda_i(D^2 u) = 0, \quad (3.6)$$

where $\alpha_1, \dots, \alpha_n \geq 0$, $\min(\alpha_1, \alpha_n) > 0$, and $\sum_{j=1}^n \alpha_j = 1$ and $\lambda_1 \leq \dots \leq \lambda_n$ are the eigenvalues of the Hessian (cf. [FV20, FG22]).

Games for convex combinations of eigenvalues

We take into account the following DPP

$$u_\varepsilon(x) = \sum_{j=1}^n \alpha_j \inf_{\dim(S)=j} \sup_{\substack{v \in S \\ |v|=1}} \frac{u_\varepsilon(x + \varepsilon v) + u_\varepsilon(x - \varepsilon v)}{2} \quad (3.7)$$

which is associated with (3.6).

Local Hölder regularity

- ▶ Ferrari and Galise [FG22] derived Hölder regularity for solutions to (3.6) with the exponent $\gamma \in (0, \delta]$ for $\delta = 1 - \frac{\alpha_1 + \alpha_n}{(\sqrt{\alpha_1} + \sqrt{\alpha_n})^2}$.
 - $0 < \delta \leq \frac{1}{2}$, and the equality only holds when $\alpha_1 = \alpha_n$.
- ▶ Blanc, H., Parviainen and Ruosteenoja [BHP23] derived Hölder regularity for solutions to (3.7) and (3.6) with the exponent $\gamma \in (0, \delta)$ for $\delta = \frac{1}{2}$.
 - It is crucial to investigate (3.7) for the case $\alpha_1 = \alpha_n = \frac{1}{2}$ (i.e. $\alpha_2 = \dots = \alpha_{n-1} = 0$).

Thank you for your attention!

References



Á. Arroyo, P. Blanc, and M. Parviainen.

Krylov-safonov theory for pucci-type extremal inequalities on random data clouds.
arXiv preprint arXiv:2410.01642, 2024.



Á. Arroyo, H. Luiro, M. Parviainen, and E. Ruosteenoja.

Asymptotic Lipschitz regularity for tug-of-war games with varying probabilities.
Potential Anal., 53(2):565–589, 2020.



P. Blanc, F. Charro, J. J. Manfredi, and J. D. Rossi.

Games associated with products of eigenvalues of the Hessian.
Math. Eng., 5(3):Paper No. 066, 26, 2023.



P. Blanc, J. Han, M. Parviainen, and E. Ruosteenoja.

Game-theoretic approach to Hölder regularity for PDEs involving eigenvalues of the Hessian.
Potential Anal., 59(4):1995–2015, 2023.



P. Blanc and J. D. Rossi.

Games for eigenvalues of the Hessian and concave/convex envelopes.
J. Math. Pures Appl. (9), 127:192–215, 2019.



F. Ferrari and G. Galise.

A regularity result for a class of non-uniformly elliptic operators.
Arch. Math. (Basel), 118(5):539–548, 2022.

References



F. Ferrari and A. Vitolo.

Regularity properties for a class of non-uniformly elliptic Isaacs operators.
Adv. Nonlinear Stud., 20(1):213–241, 2020.



J. Han.

Local Lipschitz regularity for functions satisfying a time-dependent dynamic programming principle.
Commun. Pure Appl. Anal., 19(5):2617–2640, 2020.



J. Han.

Time-dependent tug-of-war games and normalized parabolic p -Laplace equations.
Nonlinear Anal., 214:Paper No. 112542, 23, 2022.



J. Han.

Tug-of-war games related to p -laplace type equations with lower order terms.
arXiv preprint arXiv:2506.07537, 2025.



J. Han.

Tug-of-war games related to oblique derivative boundary value problems with the normalized p -Laplacian.
Potential Anal., 63(2):947–981, 2025.



J. Han and H. Liu.

Consistency of tug-of-war type operators on random data clouds.
arXiv preprint arXiv:2507.18383, 2025.

References



J. J. Manfredi, M. Parviainen, and J. D. Rossi.

On the definition and properties of p -harmonious functions.

Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 11(2):215–241, 2012.



Y. Peres and S. Sheffield.

Tug-of-war with noise: a game-theoretic view of the p -Laplacian.

Duke Math. J., 145(1):91–120, 2008.



Y. Peres, O. Schramm, S. Sheffield, and D. B. Wilson.

Tug-of-war and the infinity Laplacian.

J. Amer. Math. Soc., 22(1):167–210, 2009.