MATH6350 Homological Algebra

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Contents

Course Information

Textbooks:

- Homological Algebra, Cartan and Eilenberg.
- Methods of Homological Algebra, Sergei I. Gelfand, Yuri I. Manin.

Website: MATH6530 Homological Algebra

Previous Lecture Notes:

- Homological Algebra
- Homotopical algebra and algebraic K-theory
- DG algebras and DG categories

There will be weekly homework (exercises or problems). Students are encouraged to do the homework, although the homework will not be graded.

Tentative Plans:

1. Preliminaries
   (a) Categories and Functors
   (b) Abelian Categories (Gothendieck's Axioms)
2. Classical Homological Algebra (linear/abelian)
   (a) Chain Complexes and Basic Operations
   (b) (Abelian) Derived Functors
   This is the tool for approximation of some constructions.
   (c) Tor and Ext, Categories of Modules
   (d) Homological Dimension(s)

3. Spectral Sequence

   Based on J-P Serre. This is the main computational tool.

4. Non-abelian Homological Algebra\\

   (a) Simplicial Objects (Dold-Kan Correspondence)

   (b) Non-abelian Derived Functors

Sins of Omission: Derived and Triangulated Categories. There will be a course available next Fall.

Part I

Introduction

We shall start with the question:

> What is homological algebra?

Let's take a look at its origin, topology.

Let $f_0, f_1 : X \to Y$ be two continuous maps between two topological spaces X and Y. We say that f_0 is *homotopic* to f_1, written as $f_0 \sim f_1$ if there exists a continuous map $h : X \times [0,1] \to Y$, such that $h|_{X\times\{t\}} = f_t$ for $t = 0, 1$. This map h is called a *homotopy* between f_0 and f_1.

We can define the (classical) homotopy category $\mathbf{Ho}\left(\mathbf{Top}\right)$ of topological spaces by

$$Ob\left(\mathbf{Ho}\left(\mathbf{Top}\right)\right) = Ob\left(\mathbf{Top}\right).$$

$\mathrm{Hom}_{\mathbf{Ho}(\mathbf{Top})}\left(X, Y\right) = [X, Y] :=$the set of homotopy classes of maps from X to Y.

The iso-class of a space in $\mathbf{Ho}\left(\mathbf{Top}\right)$ is called a homotopy type.

Remark. Homotopy category is much smaller than **Top**. Passing to homotopy types we ignore local properties of spaces. In this way, we simplify spaces.

Example. The solid torus $S^1 \times D^2$ is homotopic to the circle S^1.

This point of view provides us a similar idea to understand homological algebra.

> *Homological algebra is a way to "simplify" algebraic structure (such as groups, modules, rings, etc.) in the same way homotopy theory simplifies topological spaces. It is the way of assigning "homotopy types" to each object. The goal is to study homotopy-invariant functors on algebraic objects.*

If we have a functor $F : \mathrm{Top} \to \mathcal{A}$ where $\mathcal{A}$ can be the category of abelian groups or vector spaces which perseveres homotopy types, i.e. this functor can be factorized through $\mathbf{Ho}\left(\mathbf{Top}\right)$,

$$F: \quad \begin{tikzcd} \mathbf{Top} \arrow[rr] \arrow[dr] & & \mathcal{A} \\ & \mathbf{Ho}(\mathbf{Top}) \arrow[ur] & \end{tikzcd}$$

such a functor is called a homotopic functor.

Similarly, if we have a homotopic functor $F : \mathcal{C} \to \mathcal{A}$ where $\mathcal{C}$ is some algebraic category, we shall have

$$F: \quad \begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{A} \\ & \mathbf{Ho}(\mathcal{C}) = \mathcal{D}(\mathcal{C}) \arrow[ur] & \end{tikzcd}$$

where $\mathcal{D}(\mathcal{C})$ is the derived category.

Example. The Ext and Tor functor.

Theory of Linear Differential Equations from Homological Point of View.

We shall look at the following classical example in linear PDEs to see the importance of homological algebra. We will come back to this example later in class from time to time.

Example. Let $X \subseteq \mathbb{C}^n$ be an open subset, $n \geq 1$. $\mathcal{O} = \mathcal{O}^{an}(X)$ is the commutative ring of analytic functions defined on X. $\mathcal{D} = \mathcal{D}^{an}(X)$ is the non-commutative ring (and in fact an algebra) of linear differential operators with coefficients in $\mathcal{O}$. In standard coordinates,

$$\mathcal{D} \cong \mathcal{O}\left[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right]$$

(this bracket notation is not correct because the differential operators are not commutative, but we abuse the notation for simplification) where $\frac{\partial}{\partial x_i}$ are the linear differential operators

$$\begin{array}{rccc} \frac{\partial}{\partial x_i}: & \mathcal{O} & \longrightarrow & \mathcal{O} \\ & f(x_1, \cdots, x_n) & \longmapsto & \frac{\partial f}{\partial x_i}(x_1, \cdots, x_n) \end{array}$$

Thus a linear differential operator P on X is just an element in $\mathcal{D}$ in the form

$$P(x, \partial) = \sum_{i_1, \cdots, i_n}^{N} a_{i_1} \cdots a_{i_n}(x_1, \cdots, x_n)\left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}.$$

For instance, the Laplacian operator $\Delta_n = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_n}\right)^2$.

A linear differential equation can be regarded as a kernel of such an operator,

$$Pu = 0 \tag{1}$$

where u is the unknown function, $u \in \mathcal{O}$ or $u \in C^{\infty}(X)$ or $u \in D^{1}(X)$.

A system of linear differential equations can be as a kernel of a system of operators,

$$\mathbb{P}\boldsymbol{u} = 0, \text{ or } \sum_{j=1}^{l} P_{ij}\left(x, \frac{\partial}{\partial x}\right) u_j = 0, i = 1, \cdots, k \tag{2}$$

where $\boldsymbol{u} = (u_1, \cdots, u_l)$ are unknown functions.

In general, the system makes sense for u living in any (left) $\mathcal{D}$-modules.

Given any $S \in \mathcal{D} - \text{Mod}$, we define the solution set as

$$\text{Sol}_{\mathbb{P}}(S) = \left\{\boldsymbol{u} = (u_1, \cdots, u_l) \in S^l : \mathbb{P}u = 0\right\}.$$

Definition 0.1. Two systems of type 2 are *equivalent* if for every $S \in \mathcal{D} - \text{Mod}$, the solution sets are the same, i.e.

$$\mathbb{P} \sim \mathbb{P}' \Longleftrightarrow \text{Sol}_{\mathbb{P}}(S) \cong \text{Sol}_{\mathbb{P}'}(S), \forall S \in \mathcal{D} - \text{Mod}.$$

Note that equivalent systems may have very different presentations in terms of matrix differential operations.

Example 0.1. If $n = 2, l = 1$, the following two systems of linear differential equations are equivalent.

$$\mathbb{P} : \begin{cases} \frac{\partial u}{\partial x_1} = 0 \\ \frac{\partial u}{\partial x_2} = 0 \end{cases} \quad \text{and} \quad \mathbb{P}' : \begin{cases} \frac{\partial u}{\partial x_1} = 0 \\ \frac{\partial u}{\partial x_2} = 0 \\ \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \end{cases}$$

Problem. Is there an invariant way to present systems of linear PDEs?

Definition 0.2. We can define the *solution functor* for system 2

$$\begin{array}{rrcl} \mathrm{Sol}_{\mathbb{P}}: & \mathcal{D}-\mathrm{Mod} & \longrightarrow & \mathrm{Vect}_{\mathbb{C}} \\ & S & \longmapsto & \mathrm{Sol}_{\mathbb{P}}(S) \\ & \left(S \xrightarrow{f} S'\right) & \mapsto & \left(\mathrm{Sol}_{\mathbb{P}}(S) \xrightarrow{\mathrm{Sol}_{\mathbb{P}}(f)} \mathrm{Sol}_{\mathbb{P}}(S')\right) \end{array}$$

where the map $\mathrm{Sol}_{\mathbb{P}}(S) \xrightarrow{\mathrm{Sol}_{\mathbb{P}}(f)} \mathrm{Sol}_{\mathbb{P}}(S')$ is given by $(u_1, \cdots, u_l) \mapsto (f(u_1), \cdots, f(u_l))$. Here $(f(u_1), \cdots, f(u_l))$ is a solution in $\mathrm{Sol}_{\mathbb{P}}(S')$ because f is a morphism between $\mathcal{D}$-modules and it commutes with $\mathbb{P}$.

This definition says that $\mathbb{P} \sim \mathbb{P}'$ if and only if $\mathrm{Sol}_{\mathbb{P}} \cong \mathrm{Sol}_{\mathbb{P}'}$ as functors.

Definition 0.3. A functor $F : \mathcal{C} \to \mathbf{Sets}$ is *representable* if $F \cong \mathrm{Hom}_{\mathcal{C}}(X_F, -) =: \mathfrak{h}_{X_F}$ for some X_F.

Remark 0.1. In the next chapter we will study category theory and Yoneda's lemma, and from the latter we will know that if two functors F and F' are representable, then $F \cong F'$ if and only if $X_F \cong X_{F'}$.

Claim 0.1. The functor $\mathrm{Sol}_{\mathbb{P}}$ is representable.

Proof. Define the matrix

$$\mathbb{P} = (P_{ij})_{k\times l} \in \mathbf{Mat}_{k\times l}(\mathcal{D})$$

and the vectors

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_l \end{bmatrix} \in S^l,$$

then we can rewrite $\mathbb{P}\boldsymbol{u}$ in a matrix form. Consider the map

$$\begin{array}{rrcl} \varphi: & \mathcal{D}^k & \xrightarrow{\cdot\mathbb{P}} & \mathcal{D}^l \\ & (u_1, \cdots, u_k) & \mapsto & (u_1P_{11} + \cdots, u_kP_{k1}, \cdots,) \end{array}$$

where $\mathcal{D}^k = \underbrace{\mathcal{D} \oplus \cdots \oplus \mathcal{D}}_{k}$. Notice that φ is a homomorphism of left $\mathcal{D}$-modules. Indeed, $\forall a \in \mathcal{D}$,

$$\varphi\left(au_1, \cdots, au_l\right) = \left(au_1 P_{11} + \cdots + au_k P_{k1}, \cdots\right) = a \cdot \varphi\left(u_1, \cdots, u_k\right)$$

Define $M_{\mathbb{P}} := \operatorname{Coker}(\varphi) \cong \mathcal{D}^l / \mathcal{D}^k \cdot \mathbb{P}$. When $k = l = 1$, this is $M_P = \mathcal{D}/\mathcal{D} \cdot P$. We claim that $\mathrm{Sol}_{\mathbb{P}}$ is representable via $M_{\mathbb{P}}$,

$$\mathrm{Sol}_{\mathbb{P}} \cong \mathrm{Hom}_{\mathcal{D}}\left(M_{\mathbb{P}}, -\right).$$

Consider the exact sequence

$$\mathcal{D}^k \xrightarrow{\varphi} \mathcal{D}^l \to M_{\mathbb{P}} \to 0.$$

Fix any $S \in \mathcal{D} - \mathrm{Mod}$ and apply $\mathrm{Hom}_{\mathcal{D}}(-, S)$, we get a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{D}}\left(M_{\mathbb{P}}, S\right) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}\left(\mathcal{D}^l, S\right) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathcal{D}}\left(\mathcal{D}^k, S\right) \arrow[r] \arrow[d, "\cong"] & \cdots \\
& & S^l & S^k &
\end{tikzcd}$$

The isomorphism $\mathrm{Hom}_{\mathcal{D}}\left(\mathcal{D}^l, S\right) \xrightarrow{\cong} S^l$ is given by $\left(\mathcal{D}^l \xrightarrow{f} S\right) \mapsto \left(f\left(e_1\right), \cdots, f\left(e_l\right)\right)$ where $e_1, \cdots, e_l$ is the standard basis of $\mathcal{D}^l$. Therefore

$$\mathrm{Hom}_{\mathcal{D}}\left(M_{\mathbb{P}}, S\right) \cong \ker\left(\mathbb{P} : S^l \to S^k\right) =: \mathrm{Sol}_{\mathbb{P}}(S).$$

$\square$

Moral. $\mathcal{D}$-modules of the form $M = \operatorname{Coker}\left[\mathcal{D}^k \to \mathcal{D}^l\right]$ are coherent and they are the correct way to represent systems of linear differential equations. In this language, solution sets are interpreted as $\mathrm{Hom}_{\mathcal{D}}(M, S)$.

Need of Homological Algebra

Given a $\mathcal{D}$-module M and decompose it into $M_1 \subset M \twoheadrightarrow M/M_1 =: M_2$. We will get a short exact sequence

$$0 \to M_1 \to M \to M_2 \to 0.$$

The natural question is, can we recover the information about solution of M in terms of solutions of M_1 and M_2? The answer is no, unless we do homological algebra.

Example 0.2. Let $P = A \cdot B$ be an operator where $A = x_2 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, B = \frac{\partial}{\partial x_2}$, then $P = x_2 \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial x_2^2}$. In this case, $M_A = \mathcal{D}/\mathcal{D} \cdot A, M_B = \mathcal{D}/\mathcal{D} \cdot B$ and $M_P = \mathcal{D}/\mathcal{D} \cdot P$. We have the following short exact sequence

$$0 \to M_A \to M_P \to M_B \to 0.$$

One of the main theorem of homological algebra is the existence of higher derived functors

$$\mathrm{Ext}^n_{\mathcal{D}}(M, -) : \mathcal{D} - \mathrm{Mod} \to \mathrm{Vect}_{\mathbb{C}}, \quad n = 0, 1, \cdots.$$

such that

1. $\mathrm{Ext}^0_{\mathcal{D}}(M, -) = \mathrm{Hom}_{\mathcal{D}}(M, -)$.

2. $\mathrm{Ext}^n_{\mathcal{D}}(M, -)$ are characterized by the following long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(M_2, S) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(M, S) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(M_1, S) \arrow[dll] & \\
& \mathrm{Ext}^1_{\mathcal{D}}(M_2, S) \arrow[r] & \mathrm{Ext}^1_{\mathcal{D}}(M, S) \arrow[r] & \mathrm{Ext}^1_{\mathcal{D}}(M_1, S) \arrow[r] & \cdots
\end{tikzcd}$$

Moral. To study linear differential equations, we need to introduce "higher" solution spaces in S: $\{\mathrm{Ext}^n_{\mathcal{D}}(M, S)\}_{n \geq 0}$. In fact, it is natural to organize these spaces and the homology of a chain complex $\mathbb{R}\mathrm{Hom}_{\mathcal{D}}(M, S)$,

$$\mathbb{R}\mathrm{Hom}_{\mathcal{D}}(M, -) : \mathrm{Com}(\mathcal{D} - \mathrm{Mod}) \to \mathrm{Com}(\mathrm{Vect}_k).$$

Question. Given a system 2, when does it have a finite-dimensional space of analytic solution?

Answer. When M is a holonomic $\mathcal{D}$-module, i.e. $\mathrm{Ext}^i_{\mathcal{D}}(M, \mathcal{O}) = 0, \forall i \neq n$ where $n = \dim \mathbb{C}^n$.

Part II

Preliminaries

Plans for next three lectures:

- Categories. Basic constructions.
- Functors and morphisms of functors.
- Yoneda lemma. Representable functors. Examples.
- Limits and colimits. Adjoint functors.
- Kan-extensions.

1 Categories and Functors

1.1 Categories

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

- A class of objects denoted $\mathrm{Ob}\,(\mathcal{C})$
- A class of morphisms denoted $\mathrm{Mor}\,(\mathcal{C})$

such that for all $f \in \mathrm{Mor}\,(\mathcal{C})$, there exits unique objects $X = s\,(f)$, the source of f, and $Y = t(f)$, the target of f, and write $\mathrm{Hom}_{\mathcal{C}}\,(X, Y) = \{f \in \mathrm{Mor}\,(\mathcal{C}) : s(f) = X \text{ and } t(f) = Y\}$. Equivalently, for every ordered pair (X, Y) of objects in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}\,(X, Y)$ is defined. A morphism f in $\mathrm{Hom}_{\mathcal{C}}\,(X, Y)$ is denoted $f : X \to Y$.

This data must satisfy the following axioms:

- For any ordered triple $(X, Y, Z) \in \mathrm{Ob}\,(\mathcal{C})$ there is a map called "$\circ$"

$$\begin{aligned} \circ : \quad \mathrm{Hom}_{\mathcal{C}}\,(X,Y) \times \mathrm{Hom}_{\mathcal{C}}\,(Y,Z) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}\,(X,Z) \\ \left(\left(X \xrightarrow{f} Y\right), \left(Y \xrightarrow{g} Z\right)\right) &\longmapsto \left(g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z\right) \end{aligned}$$

- For every $X \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that $\mathrm{Id}_X \circ f = f$ and $g \circ \mathrm{Id}_X = g$ for all $f : Y \to X$ and for all $g : X \to Y$.

Remark 1.1. A convenient way to organize this data is as the pullback diagram

$$\begin{tikzcd}
\mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Mor}(\mathcal{C}) \arrow[d, "s"] \\
\mathrm{Mor}(\mathcal{C}) \arrow[r, "t"] & \mathrm{Ob}(\mathcal{C})
\end{tikzcd}$$

where our data is described by

$$\mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Mor}(\mathcal{C}) \xrightarrow{\circ} \mathrm{Mor}(\mathcal{C}) \underset{\xrightarrow[s]{}}{\overset{\xrightarrow{t}}{\xleftarrow{\mathrm{Id}}}} \mathrm{Ob}(\mathcal{C}).$$

Example 1.1. Roughly, examples can be divided into three groups.

1. Categories $\mathcal{C}$ whose objects are sets with additional structure and morphisms are maps of sets preserving that structure. Examples of these categories include

   - **Set** the category of sets and set functions.
   - **Top** the category of topological spaces and continuous maps.
   - **Gr** the category of groups and group homomorphisms.
   - **Ab** the sub category of **Gr** of abelian groups.
   - **Ring** the category of rings and ring homomorphisms.
   - $\mathbf{Vect}_k$ the category of vector spaces over a field k and linear maps.
   - $A-\mathbf{Mod}$ and $\mathbf{Mod}-A$ the category of left modules and the category of right modules respectively over a ring A and module homomorphisms.
   - $\mathbf{Alg}_k$ the category of algebras over a field $k$and field homomorphisms.

2. Categories $\mathcal{C}$ whose objects are still sets with some extra structure but the morphisms are not maps of sets. Examples of these categories include

   - $\mathbf{Ho}(\mathbf{Top})$ the category of topological spaces and homotopy classes of maps.

- **Fun** $(\mathcal{C}, \mathcal{D})$ the category of functors and natural transformations.

3. Categories $\mathcal{C}$ corresponding to algebraic structures. Examples of these categories include

- **Monoids** are categories with exactly one object.
- **Groupoids** are categories in which every morphism is invertible. For example, a group considered as a one object category with its elements as morphisms is a groupoid.
- **Poset** Given a preordered set $(\mathrm{I}, \leq)$, define a category I by declaring that $\mathrm{Ob}\,(\mathbf{I}) = \mathrm{I}$ and

$$\mathrm{Hom}_{\mathbf{I}}\,(x, y) = \begin{cases} \emptyset, & x \nleq y \\ \longrightarrow, & x \leq y. \end{cases}$$

- A topological space X gives rise to a category $\mathbf{Open}(X)$ whose objects are open subsets of X and

$$\mathrm{Hom}_{\mathbf{Open}(X)}\,(U, V) = \begin{cases} \emptyset, & U \nsubseteq V \\ \longrightarrow, & U \subseteq V. \end{cases}$$

- A *quiver* is a directed graph. A quiver Q defines a category whose objects are the vertices of Q and morphisms are paths connecting vertices. The following special examples of quivers will appear later in the course

  (a) The *pullback quiver* $\{ \bullet \longrightarrow \bullet \longleftarrow \bullet \}$.

  (b) The *push-out quiver* $\{ \bullet \longleftarrow \bullet \longrightarrow \bullet \}$.

  (c) The *equalizer quiver* $\{ \bullet \rightrightarrows \bullet \}$.

Example 1.2. The *cosimplicial category*, denoted $\mathbf{\Delta}$, is defined by

$$\mathrm{Ob}\,(\mathbf{\Delta}) = \{[n] = \{0, 1, \ldots, n\}\}_{n \geq 0}$$
$$\mathrm{Hom}_{\mathbf{\Delta}}\,([n], [m]) = \{f : [n] \longrightarrow [m] : f(i) \leq f(j) \text{ if } i \leq j\}.$$

A geometric realization of the cosimplicial category is the category whose objects are geometric simplicies Δ^n. Recall that Δ^n is the convex hull of unit vectors $e_0, e_1, \ldots, e_n$ in $\mathbb{R}^{n+1}$. Morphisms between geometric simplicies are given by

$$\mathrm{Hom}_{\mathbf{\Delta}}\left(\Delta^{n},\Delta^{m}\right)=\left\{\text{linear maps } f:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}^{n+1} \text{ sending } e_i\mapsto e_{f(i)}\right\}.$$

It is worth noting that $\infty-$categories can be thought of as simplicial sets. That is, as a contravariant functor $\mathbf{\Delta}^{\mathrm{op}}\longrightarrow\mathbf{Set}$ satisfying (weak) Kan extension condition.

Definition 1.2. A category is called *small* if $\mathrm{Ob}\left(\mathcal{C}\right)$ is a set.

Remark 1.2. **Set** is not a small category. Other categories that are not small include **Gr**, **Ring, Ab**. One can enlarge the "universe" of sets and add one axiom to the standard axiomatics that requires every set X to belong to some universe $\mathcal{U}$ and declare that $X^{'}$ is $\mathcal{U}-$*small* if $X^{'}\cong X$ for some $X\in\mathcal{U}$. A precise discussion of this construction can be found in [SGA4].

Definition 1.3. A category is *locally small* if $\mathrm{Hom}_{\mathcal{C}}\left(X,Y\right)$ is a set for every $X,Y\in\mathrm{Ob}\left(\mathcal{C}\right)$.

Definition 1.4. A morphism $f:X\longrightarrow Y$ in $\mathcal{C}$ is

- an *isomorphism* if there exists a morphism $g:Y\longrightarrow X$ such that $fg=\mathrm{Id}_Y$ and $gf=\mathrm{Id}_X$.
- an *endomorphism* if $X=Y$.
- an *automorphim* if $X=Y$ and f is an isomorphism.
- a m*onomorphism* (or *monic*) if for any parallel pair: $g_1,g_2:Z\longrightarrow X$, $f\circ g_1=f\circ g_2$ implies that $g_1=g_2$. Equivalently, f is monic if the map

$$f\circ-:\mathrm{Hom}_{\mathcal{C}}\left(Z,X\right)\longrightarrow\mathrm{Hom}_{\mathcal{C}}\left(Z,X\right)$$

  is injective.
- an *epimorphism* (or epi) if for any parallel pair: $g_1,g_2:Y\longrightarrow Z$, $g_1\circ f=g_2\circ f$ implies that $g_1=g_2$. Equivalently, f is epi if the map

$$-\circ f:\mathrm{Hom}_{\mathcal{C}}\left(Y,Z\right)\longrightarrow\mathrm{Hom}_{\mathcal{C}}\left(X,Z\right)$$

  is injective.

Example 1.3. In the first class of examples from Example 1.3, monics are injective maps of the underlying sets. However, this is not the case for epimorphisms. For example, in the category **Ring** of unital rings, and the category of (Hausdorff) topological space, the map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the respective categories but is set theoretically not surjective.

Definition 1.5. An object in $\mathcal{C}$ is called *initial*, denoted $\emptyset$, if for every $X \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $f : \emptyset \longrightarrow X$. It is *terminal*, denoted $\star$, if for every $X \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $f : X \longrightarrow \star$. It is *null* if $\emptyset = \star$.

Example 1.4. In **Set**, the empty set is an initial object, any one-element set is a terminal object, and there is no terminal object. In **Gr** the trivial group is the null object.

Remark 1.3. (Due to Quillen) In the theory of model categories, it was discovered that the existence of null objects make homotopy theory much richer.

1.2 Functors

Given two categories $\mathcal{C}$ and $\mathcal{D}$.

Definition 1.6. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of

- a map

$$\begin{array}{rrcl} F: & Ob(\mathcal{C}) & \longrightarrow & Ob(\mathcal{D}) \\ & X & \longmapsto & FX \end{array}$$

- a family of maps, one for each pair $(X, Y) \in Ob(\mathcal{C}) \times Ob(\mathcal{C})$

$$\begin{array}{rrcl} F: & \mathrm{Hom}_{\mathcal{C}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(FX, FY) \\ & \left(X \xrightarrow{f} Y\right) & \longmapsto & \left(FX \xrightarrow{Ff} FY\right) \end{array}$$

  satisfying

  1. $F(\mathrm{Id}_X) = \mathrm{Id}_{FX}, \forall X \in Ob(\mathcal{C})$.
  2. F$(g \circ f) = Fg \circ Ff$ wherever f and g are compatible maps in $\mathcal{C}$.

Definition 1.7. A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{op} \to \mathcal{D}$ where $\mathcal{C}^{op}$ is the opposite category defined by

- Objects: $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ (write $X^o \in Ob(\mathcal{C}^{op})$ for $X \in Ob(\mathcal{C})$), and
- Morphisms: $\mathrm{Hom}_{\mathcal{C}^{op}}(X^o, Y^o) \coloneqq \mathrm{Hom}_{\mathcal{C}}(Y, X)$ for any $X, Y \in Ob(\mathcal{C})$.

Example 1.5. (Contravariant Functors)

1. $\mathrm{op} : \mathcal{C} \to \mathcal{C}^{op}$, this is the identity functor $\mathcal{C}^{op} \to \mathcal{C}^{op}$.
2. Duality $* : \mathrm{Vect}_k^{op} \to \mathrm{Vect}_k, V \mapsto V^* = \mathrm{Hom}_k(V, k)$.

Definition 1.8. A functor $F : \mathcal{C} \to \mathcal{D}$ is called

- *faithful* if $\forall X, Y \in Ob(\mathcal{C})$, $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is injective.
- *full* if $\forall X, Y \in Ob(\mathcal{C})$, $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is surjective.
- *fully faithful* if if $\forall X, Y \in Ob(\mathcal{C})$, $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is both injective and surjective.
- *essentially surjective* if $\forall Y \in Ob(\mathcal{D})$, $\exists X \in Ob(\mathcal{C})$ such that $FX \cong Y$.
- *conservative* if f is an isomorphism in $\mathcal{C} \Leftrightarrow Ff$ is an isomorphism in $\mathcal{D}$.

Definition 1.9. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can define the *product* $\mathcal{C} \times \mathcal{D}$ by

- Objects: $Ob(\mathcal{C} \times \mathcal{D}) = Ob(\mathcal{C}) \times Ob(\mathcal{D})$, and
- Morphisms: $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((X_1, Y_1), (X_2, Y_2)) = \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \times \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2)$ for any $X_1, X_2 \in Ob(\mathcal{C})$ and $Y_1, Y_2 \in Ob(\mathcal{D})$.

A *bifunctor* is a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$.

Example 1.6. For any category $\mathcal{C}$ we have the Hom bifunctor

$$\begin{aligned} \mathrm{Hom} : \quad \mathcal{C}^{op} \times \mathcal{C} &\longrightarrow \mathbf{Set} \\ (X, Y) &\longmapsto \mathrm{Hom}_{\mathcal{C}}(X, Y) \end{aligned}$$

which is contravariant in the first argument and covariant in the second argument.

1.3 Morphisms of Functors (Natural Transformations)

Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors.

Definition 1.10. A *morphism of functors* $\alpha : F \Rightarrow G$ is a collection of morphisms in $\mathcal{D}$ indexed by objects in $\mathcal{C}$:

$$\alpha = \{\alpha_X : FX \to GX\}_{X \in Ob(\mathcal{C})}$$

satisfying (the natural property) that for any $f : X \to Y$ in $\mathcal{C}$, the following diagram

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"] & GY \end{tikzcd}$$

commutes in $\mathcal{D}$.

Example 1.7. For each $n \geq 1$, we have a functor

$$\begin{aligned} GL_n : \quad \mathbf{CommRing} &\longrightarrow \mathbf{Gr} \\ A &\longmapsto GL_n(A) \end{aligned}$$

where $GL_n(A)$ is the group of invertible $n \times n$ matrices with entries in A.

For $n = 1$,

$$\begin{aligned} GL_1 : \quad \mathbf{CommRing} &\longrightarrow \mathbf{Gr} \\ A &\longmapsto A^* = \{\text{units in } A\} \end{aligned}$$

We have a morphism of functors $\det_n : GL_n \to GL_1$ for each $n \geq 1$,

$$\det_n = \{(\det_n)_A : GL_n(A) \to GL_1(A), M \mapsto \det(M)\}_{A \in Ob(\mathbf{CommRing})}.$$

Compositions of Morphisms of Functors

Morphisms of functors can be composed in two ways, namely "vertical" and "horizontal" compositions of morphisms of functors.

1. "Vertical" composition of morphisms of functors.

   Given three functors $F_1, F_2, F_3 : \mathcal{C} \to \mathcal{D}$ and two morphisms of functors $\alpha : F_1 \Rightarrow F_2$ and

$\beta : F_2 \Rightarrow F_3$, we can define their composition in the following manner,

$$\beta\alpha = \left\{ (\beta\alpha)_X : F_1X \xrightarrow{\alpha_X} F_2X \xrightarrow{\beta_X} F_3X \right\}_{X \in Ob(\mathcal{C})},$$

which can be illustrated as the following graph.

$$\begin{tikzcd}
\mathcal{C}\bullet \arrow[rr, bend left=40, "F_1"] \arrow[rr, "F_2" description] \arrow[rr, bend right=40, "F_3"'] \arrow[rr, phantom, bend left=20, "\Downarrow\alpha"] \arrow[rr, phantom, bend right=20, "\Downarrow\beta"] & & \bullet\mathcal{D}
\end{tikzcd}$$

2. "Horizontal " composition of morphisms of functors.

   Given four functors $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ and $G_1, G_2 : \mathcal{D} \to \mathcal{E}$ and two morphisms of functors $\alpha : F_1 \Rightarrow F_2$ and $\beta : G_1 \Rightarrow G_2$,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}$$

   the composition is defined in the following manner,

$$\beta \circ \alpha = \{(\beta \circ \alpha)_X : G_1F_1X \to G_2F_2X\}_{X \in Ob(\mathcal{C})}$$

   where $(\beta \circ \alpha)_X : G_1F_1X \to G_2F_2X$ is given by the composition $(\beta \circ \alpha)_X = \beta_{F_2X} \circ G_1\alpha_X$.

$$\begin{tikzcd}
F_1X \arrow[r, "\alpha_X"] \arrow[d, squiggly, "G_1"'] & F_2X \arrow[d, squiggly, "G_1"] \\
G_1F_1X \arrow[r, "G_1\alpha_X"] \arrow[dr, dotted, "(\beta\circ\alpha)_X"'] & G_1F_2X \arrow[d, "\beta_{F_2X}"] \\
& G_2F_2X
\end{tikzcd}$$

Special Cases Convolutions of functors and morphisms.

1. When $F_1 = F_2 = F$ and $\alpha = \mathrm{Id}_F$, this reduces to

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}$$

and we use the notation

$$\beta \circ F = \left\{ (\beta \circ F)_X : G_1 F X \xrightarrow{\beta_{FX}} G_2 F X \right\}_{X \in Ob(\mathcal{C})}$$

2. When $G_1 = G_2 = G$ and $\beta = \mathrm{Id}_G$, this reduces to

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "G"] & \mathcal{E} \end{tikzcd}$$

and we use the notation

$$G \circ \alpha = \left\{ (G \circ \alpha)_X : G F_1 X \xrightarrow{G\alpha_X} G F_2 X \right\}_{X \in Ob(\mathcal{C})}$$

1.4 Functor Categories

Definition 1.11. Let $\mathcal{C}$ be a small category (i.e. the objects in $\mathcal{C}$ form a set) and $\mathcal{D}$ be any category. Define the functor category $\mathcal{D}^{\mathcal{C}} := \mathbf{Fun}(\mathcal{C}, \mathcal{D})$ with

- Objects: functors from $\mathcal{C}$ to $\mathcal{D}$, and
- Morphisms: morphisms of functors from $\mathcal{C}$ to $\mathcal{D}$, and
- Composition: vertical composition.

Remark 1.4. If $\mathcal{C}$ and $\mathcal{D}$ carry extra structures (e.g. additive categories) then we require the functor in $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ to preserve these structures.

Examples

Presheaves. Let X be a topological space. $\mathbf{Op}(X)$ is the category of open sets in X with

- Objects: $Ob(\mathbf{Op}(X)) = \{U \subseteq X : U \text{ is open}\}$.
- Morphisms: $\mathrm{Hom}_{\mathbf{Op}(X)}(U, V) = \begin{cases} \emptyset & U \not\subseteq V \\ \to & U \subseteq V \end{cases}$

Then $\mathbf{Presh}(X) \coloneqq \mathbf{Fun}(\mathbf{Op}(X)^{op}, \mathbf{Set})$ is the presheaves (of sets) over X, and $\mathbf{AbPresh}(X) \coloneqq \mathbf{Fun}(\mathbf{Op}(X)^{op}, \mathbf{Ab})$ is the category of abelian presheaves over X. Explicitly, an abelian presheaf $\mathcal{F}$ on X is given by an assignment of abelian groups to every open $U \subseteq X$,

$$U \longmapsto \mathcal{F}(U)$$

with group homomorphism (restriction morphism) $\rho_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ for each $U \subseteq V$, such that for any triple $U \subseteq V \subseteq W$ of open subsets in X, $\rho_U^W = \rho_U^V \circ \rho_V^W : \mathcal{F}(W) \to \mathcal{F}(U)$.

Exercise 1.1. Check that this data is precisely a functor.

Simplicial Objects. Recall that Δ is the cosimplicial category with

- Objects: $Ob(\Delta) = \{[n]\}_{n\geq 0}$ where $[n] = \{0 < 1 < 2 < \cdots < n\}$.
- Morphisms: $f : [n] \to [m]$ order-preserving maps of sets.

Definition 1.12. $\mathbf{sSet} \coloneqq \mathbf{Fun}(\Delta^{op}, \mathbf{Set})$ is the *category of simplicial sets*. More generally, if $\mathcal{C}$ is any category, we can define the *category of simplicial objects* in $\mathcal{C}$, $s\mathcal{C} = \mathbf{Fun}(\Delta^{op}, \mathcal{C})$.

The categories $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ are the categories where we apply homological algebra.

Algebraic Theories. All basic algebraic categories (e.g.$\mathbf{Gr}, \mathbf{Ring}, \mathbf{CommRing}, \mathbf{Mod}, \cdots$) i.e. categories whose objects are sets with binary operations satisfying natural axioms, can be viewed as functor categories.

Definition 1.13. An *algebraic theory* is a small category $\mathcal{T}$ equipped with finite products (see later for precise definition) and objects $\{T_n\}_{n\geq 0}$ for each natural number such that for every n, T_n is equipped with an isomorphism $T_n \xrightarrow{\cong} \underbrace{T_1 \times \cdots \times T_1}_{n}$. A $\mathcal{T}$*-algebra* (or *model* for $\mathcal{T}$) in a category $\mathcal{C}$ is a functor $A : \mathcal{T} \to \mathcal{C}$ preserving products in the sense that the natural morphisms in $\mathcal{C}$

$$A(T_n) \to \underbrace{A(T_1) \times \cdots \times A(T_1)}_{n}$$

obtained by applying A to $P_{n,i} : T_n \to T_1, i = 1, \cdots, n$ is an isomorphism. Denote by $\mathbf{Alg}_{\mathcal{T}}(\mathcal{C})$ the category of $\mathcal{T}$-algebras in $\mathcal{C}$.

Example 1.8. Let $\mathcal{C} = \mathbf{Set}$ be the category of sets. $\mathbf{Gr}$ is the category of groups. Let $\mathcal{T}$ be the full subcategory of free groups, with objects

$$F_n \coloneqq \mathbb{F}\langle x_1, \cdots, x_n\rangle, n \geq 1$$
$$F_0 \coloneqq e$$

Note that there is a natural isomorphism

$$\underbrace{F_1 \sqcup \cdots \sqcup F_1}_{n} \xrightarrow{\cong} F_n \quad \text{(coproduct)}$$

Let $\mathcal{F} = \mathcal{T}^{op}$, then we claim that $\mathbf{Alg}_{\mathcal{F}}(\mathbf{Set}) \cong \mathbf{Gr}$.

Exercise 1.2. Prove $\mathbf{Alg}_{\mathcal{F}}(\mathbf{Set}) \cong \mathbf{Gr}$.

Example 1.9. $\mathbf{Fin}$ is the category of finite sets with objects $\underline{n} = \{1, \cdots, n\}$ and $\underline{0} = \emptyset$. Note that for all $n \in \mathbb{N}$, there is a natural isomorphism

$$\underbrace{\underline{1} \sqcup \cdots \sqcup \underline{1}}_{n} \xrightarrow{\cong} \underline{n}.$$

Take $\mathcal{T} = \mathbf{Fin}^{op}$, this is the algebraic theory of commutative unital k-algebras, i.e. $\mathbf{Alg}_{\mathbf{Fin}^{op}}(\mathbf{Vect}_k) \cong \mathbf{CommAlg}_k$.

Exercise 1.3. Prove $\mathbf{Alg}_{\mathbf{Fin}^{op}}(\mathbf{Vect}_k) \cong \mathbf{CommAlg}_k$.

The Center of A Category

In the definition of a functor category, consider the case when $\mathcal{C} = \mathcal{D}$ and denote the identity functor on $\mathcal{C}$ by $\mathrm{Id}_{\mathcal{C}}$. The *center of a category* $\mathcal{C}$, denoted $\mathcal{Z}(\mathcal{C})$, is defined as

$$\mathcal{Z}(\mathcal{C}) = \mathrm{End}(\mathrm{Id}_{\mathcal{C}}) := \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}}).$$

By definition, $\mathcal{Z}(\mathcal{C})$ is an associative unital semigroup (i.e. a monoid). One can also consider $\mathrm{Aut}(\mathrm{Id}_{\mathcal{C}})$ which is the group of isomorphisms of the identity functor.

Lemma 1.1. *$\mathcal{Z}(\mathcal{C})$ is a commutative monoid. Consequently, $Aut(Id_{\mathcal{C}})$ is an abelian group.*

Proof. By definition, a morphism $\alpha : \mathrm{Id}_{\mathcal{C}} \Longrightarrow \mathrm{Id}_{\mathcal{C}}$ is given by

$$\alpha = \{\alpha_X : X \to X\}_{X \in Ob(\mathcal{C})}$$

such that for any $f : X \to Y$ in $\mathcal{C}$, the following diagram

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\alpha_Y"] & Y \end{tikzcd} \tag{3}$$

commutes. Given $\alpha, \beta \in \mathcal{Z}(\mathcal{C})$, consider the diagram (3) for α with $Y = X$ and $f = \beta_X$. Then the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "\beta_X"'] & X \arrow[d, "\beta_X"] \\ X \arrow[r, "\alpha_X"] & X \end{tikzcd}$$

implies that $\alpha\beta = \beta\alpha$. $\square$

One interesting example is the center of a module category.

Theorem 1.1. *Let R be an associative unital ring and $R-\mathbf{Mod}$ the category of left modules over R. There is a natural isomorphism $\mathcal{Z}(R-\mathbf{Mod}) \cong \mathcal{Z}(R)$, where $\mathcal{Z}(R)$ is the center of the ring. In particular, if R is commutative, then $\mathcal{Z}(R-\mathbf{Mod}) \cong R$.*

The idea of this theorem is that commutative rings can be recovered from their module (or representation) category.

Proof. We construct two maps which will turn out to be natural inverses:

$$\varphi : \mathcal{Z}(R) \longrightarrow \mathcal{Z}(R-\mathbf{Mod}) \tag{4}$$

Fix $z \in \mathcal{Z}(R)$ and define $\varphi(z) : \mathrm{Id}_{\mathcal{C}} \Rightarrow \mathrm{Id}_{\mathcal{C}}$ by

$$\varphi(z) = \left\{ \begin{array}{cccc} (\varphi(z))_M : & M & \longrightarrow & M \\ & m & \longmapsto & zm \end{array} \right\}_{M \in Ob(R-\mathbf{Mod})}.$$

Note that $\varphi(z)_M$ is well defined because for all $r \in R$ and for all $m \in M$, $\varphi(z)_M(rm) = z(rm) = (zr)m = (rz)m = r(zm)$. To see that $\varphi(z)$ is well-defined as a morphism between the identity functor, we need verify that for every $f : M \longrightarrow N$ in $R-\mathbf{Mod}$, the diagram

$$\begin{tikzcd} M \arrow[r, "(\varphi(z))_M"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "(\varphi(z))_N"] & N \end{tikzcd}$$

commutes. Indeed, for every $m \in M$, $f((\varphi(z))_M(m)) = f(zm) = zf(m) = (\varphi(z))_N(f(m))$.

$$\psi : \mathcal{Z}(R-\mathbf{Mod}) \longrightarrow \mathcal{Z}(R) \tag{5}$$

Fix $\alpha : \mathrm{Id}_{\mathcal{C}} \Rightarrow \mathrm{Id}_{\mathcal{C}}$ given by $\alpha = \{\alpha_M : M \to M\}_{M \in Ob(R-\mathbf{Mod})}$ and consider $\alpha_R : R \longrightarrow R$. Define $\psi(\alpha) = \alpha_R(1_R) \in R$. We will show that indeed $\alpha_R(1_R) \in \mathcal{Z}(R)$. Fix any $r \in R$ and define $f : R \longrightarrow R$ by $x \mapsto xr$. Since f is a morphism in $R-\mathbf{Mod}$, the diagram

$$\begin{tikzcd} R \arrow[r, "\alpha_R"] \arrow[d, "f"'] & R \arrow[d, "f"] \\ R \arrow[r, "\alpha_R"] & R \end{tikzcd}$$

commutes. In particular, it implies that $\alpha_R(1_R)r = f(\alpha_R(1_R)) = \alpha_R(f(1_R)) = \alpha_R(r) = r\alpha_R(1_R)$. Thus the map ψ is well defined. One can easily show that φ and ψ are inverse to each other. □

Remark. The category $\mathcal{C} = R-\mathbf{Mod}$ discussed in this example is an *additive category* so each $\mathrm{Hom}_R(M, N)$ is an abelian group. This can be used to define an additive structure on $\mathcal{Z}(\mathcal{C})$ by $(\alpha + \beta)_M = \alpha_M + \beta_M$ for every $M \in Ob(\mathcal{C})$.

1.5 Equivalence of Categories

It seems natural to say that a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an *isomorphism of categories* if there exists a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $G \circ F = \mathrm{Id}_{\mathcal{C}}$ and $F \circ G = \mathrm{Id}_{\mathcal{D}}$. However, this is a useless notion in mathematical practice. Unless $\mathcal{C} = \mathcal{D}$ and $F = \mathrm{Id}_{\mathcal{C}}$, this notion does not occur in nature.

Remark. (Due to V. Voevodski) In Type theory, categories viewed up to isomorphism are called *pre-categories.*

The correct notion of equivalence of categories is based on the notion of an *isomorphism of functors.*

Definition 1.14. Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors. A morphism of functors $\alpha : F \Longrightarrow G$ is an isomorphism if there exists $\beta : G \Longrightarrow F$ such that $\alpha\beta = \mathrm{Id}_G$ and $\beta\alpha = \mathrm{Id}_F$, where $\mathrm{Id}_F : F \Longrightarrow F$ is given by $(\mathrm{Id}_F)_X = \mathrm{Id}_{FX}$ for every $X \in Ob(\mathcal{C})$.

Lemma 1.2. *A morphism $\alpha : F \Longrightarrow G$ is an isomorphism if and only if for any $X \in Ob(\mathcal{C})$, $\alpha_X : FX \longrightarrow GX$ is an isomorphism in $\mathcal{D}$.*

Proof. " $\Longrightarrow$ " If $\alpha : F \Rightarrow G$ is an isomorphism, there exists $\beta : G \to F$ such that $\beta\alpha = \mathrm{Id}_F$ and $\alpha\beta = \mathrm{Id}_G$, i.e. $\beta_X\alpha_X = \mathrm{Id}_{FX}$ and $\alpha_X\beta_X = \mathrm{Id}_{GX}$ for any $X \in Ob(\mathcal{C})$. In particular, α_X is an isomorphism in $\mathcal{D}$ for any $X \in Ob(\mathcal{C})$.

" $\Longleftarrow$ " If for any $X \in Ob(\mathcal{C})$, $\alpha_X : F(X) \longrightarrow G(X)$ is an isomorphism in $\mathcal{D}$, let $\beta_X = \alpha_X^{-1} : GX \to FX$. We need to check that $\beta : G \to F$ is a morphism of functors. Given $f : X \to Y$, the diagram

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"] & GY \end{tikzcd}$$

commutes, which implies that

$$\begin{tikzcd} GX \arrow[r, "\beta_X"] \arrow[d, "Gf"'] & FX \arrow[d, "Ff"] \\ GY \arrow[r, "\beta Y"] & FY \end{tikzcd}$$

commutes, whence β is a morphism of functors. □

Exercise 1.4. Prove the lemma.

Definition 1.15. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called an *equivalence of categories* if there exists a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ together with isomorphisms of functors such that $F \circ G \cong \mathrm{Id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent,* written $\mathcal{C} \xrightarrow{\sim} \mathcal{D}$, if there exists an equivalence of categories $F : \mathcal{C} \longrightarrow \mathcal{D}$.

Remark 1.5. The functor G in the definition is far from being canonical and is said to be *quasi-inverse* to F.

Example 1.10. In the category $\mathbf{Vect}_k$ of vector spaces and linear maps over a fixed field k, consider the two *full* subcategories:

$\mathcal{C} = \{k^n\}$, the one-object subcategory with all maps $k^n \longrightarrow k^n$ which are given by $n \times n$-matrices over k.

$\mathcal{D} = \mathbf{Vect}_k^n$, the category of all $n-$dimensional vector spaces over k and linear maps between them.

There is a natural inclusion functor $i : \mathcal{C} \longrightarrow \mathcal{D}$ which is an equivalence of categories. To construct a quasi-inverse, $G : \mathbf{Vect}_k \longrightarrow \{k^n\}$ we have to choose a basis in each $V \in Ob\left(\mathbf{Vect}_k\right)$ which gives an identification $V \cong k^n$ and for $f : V \longrightarrow W$, the morphism $G\left(f\right)$ is the matrix of f in the chosen basis.

The following theorem due to Freud gives a very useful characterization of categories.

Theorem 1.2. *A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

The proof is based on the following lemma:

Lemma 1.3. *Let $\mathcal{C}$ be a category. Then there is a full subcategory $\mathcal{C}_0$ such that the inclusion functor $i : \mathcal{C}_0 \longrightarrow \mathcal{C}$ is an equivalence of categories and has the property that any two isomorphic objects in $\mathcal{C}_0$ are equal.*

Definition 1.16. Such a category is called the *skeleton* of $\mathcal{C}$ and is denoted $sk\left(\mathcal{C}\right)$.

The category $\mathcal{C}_0$ is defined by choosing exactly one object in each isomorphism class of objects in $\mathcal{C}$.

For example, $\{k^n\} = sk\left(\mathbf{Vect}_k^n\right)$ or more generally, $\left\{0, k, k^2, \ldots, k^n, \ldots\right\} = sk\left(\mathbf{Vect}_k\right)$.

Fundamental groupoids. Recall that a groupoid is a (small) category in which every morphism is an isomorphism. In particular, a groupoid with one object is a group, that is, if $\mathcal{G}$ is a groupoid with one object $\star$, then $G = \mathrm{Hom}_{\mathcal{G}}\left(\star, \star\right)$ is a group.

For a topological space X, we define its fundamental groupoid $\Pi\left(X\right)$ by $Ob\left(\Pi\left(X\right)\right) = X$ and for $x, y \in Ob\left(\Pi\left(X\right)\right)$, $\mathrm{Hom}_{\Pi(X)}\left(x, y\right)$ is the set of homotopy equivalent paths from x to y. For $x \in Ob\left(\Pi\left(X\right)\right)$, $\mathrm{End}\left(x\right) = \Pi_1\left(X, x\right)$, is the usual fundamental group of X based at x and

$\Pi_0(\Pi(X))$ is the set of connected components of X. For a general category $\mathcal{C}$, we define the set of components of $\mathcal{C}$, denoted $\Pi_0(\mathcal{C})$ by

$$\Pi_0(\mathcal{C}) = Ob(\mathcal{C}) / \sim$$

where $\sim$ is the smallest equivalence relation generated by elementary relations: $x \sim y$ if and only if there exists $x \xrightarrow{f} y$ in $\mathcal{C}$.

1.6 Representable Functors and Yoneda Lemma

Recall if X is a topological space, we define $\mathbf{Op}(X)$ the category of open sets in X and presheaves on X as functors $\mathbf{Op}(X)^{op} \to \mathbf{Set}$. We want to extend this definition to general categories.

Fix a (locally small) category $\mathcal{C}$, define $\hat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Set})$. By analogy with topology, we call $F \in Ob\left(\hat{\mathcal{C}}\right)$ a presheaf on $\mathcal{C}$.

Remark 1.6. If $\mathcal{C} = \{\star\}$ is the "pointed" category, then $\hat{\mathcal{C}} \cong \mathbf{Set}$. (Another point of view) If $\mathcal{C}$ is a small category, this suggests to think (and call) $F \in Ob\left(\hat{\mathcal{C}}\right)$ $\mathcal{C}$-sets. In fact, the category $\hat{\mathcal{C}}$ of $\mathcal{C}$-sets shows many good properties with $\mathbf{Set}$. (In fact, it is a topos.)

Given $X \in Ob(\mathcal{C})$, we can define

$$\begin{array}{rrcl} h_X : & \mathcal{C}^{op} & \longrightarrow & \mathbf{Set} \\ & Y & \longmapsto & \mathcal{C}(Y,X) := \mathrm{Hom}_{\mathcal{C}}(Y,X) \\ & \left(Y \xrightarrow{f} Z\right) & \longmapsto & \left(\begin{array}{rrcl} f^* : & \mathcal{C}(Z,X) & \longrightarrow & \mathcal{C}(Y,X) \\ & \left(Z \xrightarrow{g} X\right) & \longmapsto & \left(Y \xrightarrow{f} Z \xrightarrow{g} X\right) \end{array} \right) \end{array}$$

where f^* is the precomposition with f.

Definition 1.17. A functor (a presheaf) $F : \mathcal{C}^{op} \to \mathbf{Set}$ is *representable* if there exists $X \in Ob(\mathcal{C})$ together with isomorphism of functors $\Psi : h_X \overset{\sim}{\Rightarrow} F$ in $\hat{\mathcal{C}}$. The assignment $X \mapsto h_X$ extends to a (covariant) functor

$$\begin{array}{rrcl} h : & \mathcal{C} & \longrightarrow & \hat{\mathcal{C}} \\ & X & \longmapsto & h_X \end{array}$$

which is called the *Yoneda functor* and defined (on morphisms) as follows.

For $f : X_1 \to X_2$, define $h_f : h_{X_1} \to h_{X_2}$ by

$$h_f = \left\{ \begin{array}{rrcl} h_f(Y): & h_{X_1}(Y) = \mathcal{C}(Y, X_1) & \longrightarrow & h_{X_2}(Y) = \mathcal{C}(Y, X_2) \\ & \left(Y \xrightarrow{g} X_1\right) & \longmapsto & \left(Y \xrightarrow{g} X_1 \xrightarrow{f} X_2\right) \end{array} \right\}$$

We need to check that h_f is a morphism of contravariant functors. The following diagram

$$\begin{tikzcd} h_{X_1}(Y) \arrow[r, "h_f(Y)"] \arrow[d, "s^*"'] & h_{X_2}(Y) \arrow[d, "s^*"] \\ h_{X_1}(Z) \arrow[r, "h_f(Z)"] & h_{X_2}(Z) \end{tikzcd}$$

$$\begin{tikzcd} \left(Y \xrightarrow{g} X_1\right) \arrow[r, mapsto] \arrow[d, mapsto] & \left(Y \xrightarrow{g} X_1 \xrightarrow{f} X_2\right) \arrow[d, mapsto] \\ \left(Z \xrightarrow{s} Y \xrightarrow{g} X_1\right) \arrow[r, mapsto] & \left(Z \xrightarrow{s} Y \xrightarrow{g} X_1 \xrightarrow{f} X_2\right) \end{tikzcd}$$

commutes for any $s : Z \to Y$ because associativity of composition of morphisms, $f \circ (g \circ s) = (f \circ g) \circ s$.

It is easy to check that $h_{f \circ g} = h_f \circ h_g$ because this is equivalent to $(f \circ g)_* = f_* \circ g_*$. Thus we have a natural functor

$$\begin{array}{rrcl} h: & \mathcal{C} & \longrightarrow & \hat{\mathcal{C}} \\ & X & \longmapsto & h_X \\ & \left(X_1 \xrightarrow{f} X_2\right) & \longmapsto & h_f = \{h_f(Y) : h_{X_1}(Y) \to h_{X_2}(Y)\}_{Y \in Ob(\mathcal{C})} \end{array}$$

Theorem 1.3. *(Yoneda Lemma) For any $X \in Ob(\mathcal{C})$ and $F \in Ob\left(\hat{\mathcal{C}}\right)$ the map*

$$\begin{array}{rrcl} \psi: & Hom_{\hat{\mathcal{C}}}(h_X, F) & \longrightarrow & FX \\ & \varphi = \{\varphi_Y : h_X(Y) \to FY\}_{Y \in Ob(\mathcal{C})} & \longmapsto & \varphi_X(Id_X) \end{array}$$

is a bijection of sets.

Proof. We need to construct the inverse map, given by

$$\begin{array}{rccc} \phi: & FX & \longrightarrow & \operatorname{Hom}_{\hat{\mathcal{C}}}(h_X, F) \\ & x & \longmapsto & \phi(x) = \left\{ \begin{array}{rccc} \phi(x)_Y: & h_X(Y) & \to & FY \\ & \left(Y \xrightarrow{f} X\right) & \mapsto & Ff(x) \end{array} \right\}_{Y \in Ob(\mathcal{C})} \end{array}$$

There are three things to be checked.

1. For every $x \in FX$, $\phi(x)$ is a morphism of functors. This can be shown via the following commutative diagram

$$\begin{tikzcd} h_X(Y) \arrow[r, "\phi(x)_Y"] \arrow[d, "g^*"'] & FY \arrow[d, "Fg"] \\ h_X(Z) \arrow[r, "\phi_X(Z)"] & FZ \end{tikzcd}$$

$$\begin{tikzcd} \left(Y \xrightarrow{f} X\right) \arrow[r, mapsto] \arrow[d, mapsto] & Ff(x) \arrow[d, mapsto] \\ \left(Z \xrightarrow{g} Y \xrightarrow{f} X\right) \arrow[r, mapsto] & F(g^*f)(x) = F(f \circ g)(x) = Fg(Ff(x)) \end{tikzcd}$$

for any $g: Z \to Y$.

2. $\psi \circ \phi = \mathrm{Id}_{FX}$.

$$\psi \circ \phi(x) = \phi(x)_X(\mathrm{Id}_X) = \mathrm{Id}_X(x) = x.$$

3. $\phi \circ \psi = \mathrm{Id}_{\hat{\mathcal{C}}(h_X, F)}$.

$$\phi \circ \psi(\varphi) = \phi(\varphi_X(\mathrm{Id}_X)) = \left\{ \begin{array}{rccc} \phi(\varphi_X(\mathrm{Id}_X))_Y: & h_X(Y) & \to & FY \\ & \left(Y \xrightarrow{f} X\right) & \mapsto & Ff(\varphi_X(\mathrm{Id}_X)) \end{array} \right\}$$

By the commuting diagram

$$\begin{tikzcd} h_X(X) \arrow[r, "\varphi_X"] \arrow[d, "f^*"'] & FX \arrow[d, "Ff"] \\ h_X(Y) \arrow[r, "\varphi_Y"] & FY \end{tikzcd}$$

$$\begin{tikzcd} \left(X \xrightarrow{\mathrm{Id}_X} X\right) \arrow[r, mapsto] \arrow[d, mapsto] & \varphi_X\left(\mathrm{Id}_X\right) = \varphi_X\left(\mathrm{Id}_X\right) \arrow[d, mapsto] \\ \left(Y \xrightarrow{f} X\right) \arrow[r, mapsto] & \varphi_Y\left(f\right) = Ff\left(\varphi_X\left(\mathrm{Id}_X\right)\right) \end{tikzcd}$$

we see that $\phi \circ \psi\left(\varphi\right) = \varphi$.

$\square$

Corollary 1.1. *(Yoneda Lemma II) The functor $h : \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$ is fully faithful. Thus $\mathcal{C}$ can be identified with the full subcategory of $\hat{\mathcal{C}}$ consisting of representable functors. (That is, $\hat{\mathcal{C}}$ is a very natural enlargement of $\mathcal{C}$.)*

Proof. We need to show that $h : \mathrm{Hom}_{\mathcal{C}}\left(X, Y\right) \to \mathrm{Hom}_{\hat{\mathcal{C}}}\left(h_X, h_Y\right)$ is a bijection. Take $F = h_Y$ in the previous theorem,

$$\begin{array}{ccc} \mathrm{Hom}_{\hat{\mathcal{C}}}\left(h_X, h_Y\right) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}\left(X, Y\right) \\ \varphi & \longmapsto & \varphi_X\left(\mathrm{Id}_X\right) \end{array}$$

is the inverse of h for any $Y \in Ob\left(\mathcal{C}\right)$. $\square$

The presheaves on $\mathcal{C}$ that lie in the essential image of $h$are the representable functors. The functor h is called the Yoneda embedding.

Remark 1.7. Yoneda lemma implies

1. If $F : \mathcal{C}^{op} \to \mathbf{Set}$ is representable by $X \in Ob\left(\mathcal{C}\right)$, i.e. there exists $\psi : h_X \overset{\sim}{\Rightarrow} F$, then X is determined up to a canonical isomorphism. Indeed, suppose X and $X' \in Ob\left(\mathcal{C}\right)$ represent F, then there exists $\psi : h_X \overset{\sim}{\Rightarrow} F$ and $\psi' : h_{X'} \overset{\sim}{\Rightarrow} F$, then $\left(\psi'\right)^{-1} \circ \psi : h_X \overset{\sim}{\Rightarrow} h_{X'}$. Apply h^{-1} to $\left(\psi'\right)^{-1} \circ \psi$ we get $h^{-1}\left(\left(\psi'\right)^{-1} \circ \psi\right) : X \xrightarrow{\sim} X'$.

2. We can "rigidify" representability in the following manner. If F is representable with $\psi : h_X \overset{\sim}{\Rightarrow} F$ for some $X \in Ob\left(\mathcal{C}\right)$, say that F is represented by the pair $\left(X, \sigma\right)$ where $\sigma \coloneqq \psi_X\left(\mathrm{Id}_X\right) \in FX$. We call σ a *universal object*. Then $\left(X, \sigma\right)$ is determined by F uniquely up to unique isomorphism, i.e. if there exists $\left(X, \sigma\right)$ and $\left(X', \sigma'\right)$ representing F, then there exists a unique $f : X \xrightarrow{\sim} X'$ such that $Ff\left(\sigma\right) = \sigma'$.

3. We can dualize everything and consider the category $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$ and introduce for any $X \in Ob(\mathcal{C})$ a covariant functor

$$h^X : \mathcal{C} \to \mathbf{Set}, Y \mapsto \mathcal{C}(X, Y)$$

and we say that $F : \mathcal{C} \to \mathbf{Set}$ is corepresentable if there exists $X \in Ob(\mathcal{C})$ such that $F \simeq h^X$ in $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$. There is an obvious dual of Yoneda Lemma.

$$h^{\bullet} : \mathcal{C}^{op} \to \mathbf{Fun}(\mathcal{C}, \mathbf{Set})$$

is a fully faithful contravariant functor.

Examples

$\mathcal{D}$-modules (PDEs).

Example. Let $X \subseteq \mathbb{C}^n$ be an open subset, $n \geq 1$. $\mathcal{O} = \mathcal{O}^{an}(X)$ is the commutative ring of analytic functions defined on X. $\mathcal{D} = \mathcal{D}^{an}(X)$ is the non-commutative ring (and in fact an algebra) of linear differential operators with coefficients in $\mathcal{O}$. A linear differential operator P on X is just an element in $\mathcal{D}$ in the form

$$P(x, \partial) = \sum_{i_1, \cdots, i_n}^{N} a_{i_1} \cdots a_{i_n}(x_1, \cdots, x_n) \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}$$

where $\frac{\partial}{\partial x_i}$ are the linear differential operators

$$\begin{array}{rrcl} \frac{\partial}{\partial x_i}: & \mathcal{O} & \longrightarrow & \mathcal{O} \\ & f(x_1, \cdots, x_n) & \longmapsto & \frac{\partial f}{\partial x_i}(x_1, \cdots, x_n) \end{array}$$

A linear differential equation can be regarded as a kernel of such an operator $P(x, \partial) u = 0$. $\mathcal{D}$ is the ring of analytic differential operators generated as a subring of $\mathrm{End}_{\mathbb{C}}(\mathcal{O})$ by $P(x, \partial)$. Consider the solution functor

$$\begin{array}{rrcl} \mathrm{Sol}_P: & \mathcal{D}\text{-Mod} & \longrightarrow & \mathbf{Set} \\ & M & \longmapsto & \mathrm{Sol}_P(M) = \{u \in M : P(x, \partial) u = 0\} \end{array}$$

This functor is corepresentable by the left $\mathcal{D}$-module $M_P = \mathcal{D}/\mathcal{D} \cdot P$,

$$\mathrm{Sol}_P(M) \simeq \mathrm{Hom}_{\mathcal{D}\text{-Mod}}(M_P, M)$$

and $P \sim P'$ if and only if $M_P \cong M_{P'}$.

Representation Schemes. The fundamental problem in representation theory is to understand the structure of finite dimensional representations of a given (associative or Lie) algebra A defined over a field k. For instance, let Γ be a finite group and $\bar{k} = k, char(k) = 0$, then any n-dimensional linear representation of Γ is given by $k[\Gamma] \to \mathbb{M}_n(k)$. More precisely, fix $n \in \mathbb{N}$, the set of all representations of a given algebra A consists of k-algebra maps

$$\rho : A \to \mathbb{M}_n(k).$$

The natural approach is to define the representation functor

$$\begin{array}{rrcl} \mathrm{Rep}_n(A): & \mathbf{CommAlg}_k & \longrightarrow & \mathbf{Set} \\ & B & \longmapsto & \mathrm{Hom}_{\mathbf{Alg}_k}(A, \mathbb{M}_n(B)) \end{array}$$

where $\mathrm{Hom}_{\mathbf{Alg}_k}(A, \mathbb{M}_n(B))$ is the set of families of representations of A parametrized by $\mathrm{Spec}(B)$.

Note $\mathrm{Rep}_n(A)(k)$ is the set of all k-linear n-dimensional representations of A.

Proposition 1.1. *The functor $Rep_n(A)$ is corepresentable,that is, there exists a commutative algebra A_n such that*

$$Hom_{\mathbf{CommAlg}_k}(A_n, B) \cong Rep_n(A)(B) = Hom_{\mathbf{Alg}_k}(A, \mathbb{M}_n(B)),$$

with universal object the universal representation

$$\sigma : A \to \mathbb{M}_n(A_n).$$

Problem. What is A_n?

$$A_n = C/\langle [C,C] \rangle, \quad C = [A * \mathbb{M}_n(k)]^{\mathbb{M}_n(k)} .$$

where

$$C = [A * \mathbb{M}_n(k)]^{\mathbb{M}_n(k)} = \{w \in A * \mathbb{M}_n(k), [w,m] = 0, \forall m \in \mathbb{M}_n(k)\}.$$

There is an embedding

$$\begin{array}{ccc} \mathbb{M}_n(k) & \hookrightarrow & [A * \mathbb{M}_n(k)]^{\mathbb{M}_n(k)} . \\ m & \mapsto & m \end{array}$$

Recall from Algebraic Geometry that there is a natural functor

$$\begin{array}{rccc} \mathrm{Spec} : & \mathbf{CommAlg}_k^{op} & \longrightarrow & \mathbf{Schemes}_k \\ & A & \longmapsto & \mathrm{Spec}(A) \end{array}$$

that is fully faithful and whose essential image is the full subcategory of affine schemes. For example, $\mathrm{Spec}(k[a_1, \ldots, a_n]) \cong \mathbb{A}_k^n$. In particular, $\mathbf{CommAlg}_k^{op} \cong \mathbf{AffineSchemes}_k$, or more conveniently, $\mathbf{CommAlg}_k \cong \mathbf{AffineSchemes}_k^{op}$ and s

$$\begin{array}{rccc} \mathrm{Rep}_n(A) : & \mathbf{AffineSchemes}_k^{op} & \longrightarrow & \mathbf{Set} \\ & \mathrm{Spec}(B) & \longmapsto & \mathrm{Hom}_{\mathbf{Alg}_k}(A, \mathbb{M}_n(B)) . \end{array}$$

Corollary 1.2. *The functor $Rep_n(A)$ is representable by the affine scheme $Spec(A_n)$ which will be denoted $Rep_n(A)$.*

Exercise 1.5. Let k be a field and $A = k[x,y]$. Fix $n \geq 1$. Then A_n can be described explicitly as follows:

1. Consider the polynomial ring of $2n^2$ variables $k[x_{ij}, y_{ij}]_{i,j=1,2,\ldots,n}$. Define the ideal generated by n^2 relations
$$I = \left(\sum_{k=1}^{n} (x_{ik} y_{kj} - y_{ik} x_{kj}) \right)_{i,j-1,2,\ldots,n} .$$
Prove that $A_n \cong k[x_{ij}, y_{ij}]/I$.

2. Note that it is convenient to organize the variables x_{ij} and y_{ij} as matrices

$$X = \|x_{ij}\|_{i,j=1,2,\ldots,n} \quad \text{and} \quad Y = \|y_{ij}\|_{i,j=1,2,\ldots,n}$$

and notice that the relation $XY = YX = 0$ defines the ideal I. Take $B = A_n$, then to the identity element $\mathrm{Id}_{A_n} \in \mathrm{Hom}_{\mathbf{CommAlg}_k}(A_n, A_n)$, there corresponds the *universal n-dimensional representation of A*

$$\rho^{un} : A \longrightarrow \mathbb{M}_n(A_n).$$

Show that the universal representation in this case is given by

$$\begin{array}{rrcl} \rho^{un}: & k[x,y] & \longrightarrow & \mathbb{M}_n(A_n) \\ & x & \longmapsto & X \\ & y & \longmapsto & Y. \end{array}$$

Remark 1.8. $\mathrm{Rep}_n(k[x,y]) = \mathrm{Spec}(A_n)$ is called the n^{th} *commuting scheme.* For $n > 5$, and k an algebraically closed field of characteristic 0, it is an open conjecture that $\mathrm{Rep}_n(k[x,y])$ is a *reduced* scheme, namely, that the ideal I is radical. It is a well know result of Gerstenhaber (1961) that $\mathrm{Rep}_n(k[x,y])$ is irreducible.

Hilbert Schemes. Fix an algebraically closed field k of characteristic 0 and let X be a fixed projective variety over k (for instance, one can take $X = \mathbb{P}_k^n, n \geq 1$).

Problem. Classify all closed subschemes of X. That is, construct a "space" whose points are in bijection with $\mathrm{Hilb}_X(\mathrm{Spec}(k))$, the set of all closed subschemes in X.

We extend Hilb_X to a functor

$$\begin{array}{rrcl} \mathrm{Hilb}_X: & \mathbf{Schemes}_k^{op} & \longrightarrow & \mathbf{Set} \\ & U & \longmapsto & \{\text{closed subschemes } Z \subset U \times X \text{ such that } \pi_U|_Z : Z \hookrightarrow U \times X \twoheadrightarrow U \text{ is flat}\}. \end{array}$$

On morphisms, say $f : U \longrightarrow U'$, then

$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook] & Z' \arrow[d, hook] \\ U \times X \arrow[r, "f\times \mathrm{Id}_X"'] & U' \times X \end{tikzcd}$$

where $Z = (f \times \mathrm{Id}_X)^* Z'$ is the pullback of the diagram in the category of schemes.

Theorem 1.4. *(Grothendieck) The functor $Hilb_X$ is representable and the corresponding scheme is called the Hilbert scheme of X.*

Remark 1.9. Note that Hilb_X is not a variety but there is a natural stratification of Hilb_X by *honest* projective (Noetherian) schemes. The strata Hilb_X^P are indexed by *Hilbert polynomials* P which are defined as follows: For a scheme U , fix $u \in U$ and consider the scheme theoretic fiber Z_u of π_U. Define for each integer m the polynomial $P_{Z,u}(m) = \chi(\mathcal{O}_{Z_u} \otimes \mathcal{O}_X(m))$ where χ is the Euler characteristic. There is a polynomial $P_Z \in \mathbb{Z}[m]$ such that $P_Z(m) = P_{Z,u}(m)$ for all m large enough. Furthermore, P_Z is independent of U and u if U is connected. Using this, we can define the subfunctor

$$U \longmapsto \mathrm{Hilb}_X^P(U) = \{Z \in \mathrm{Hilb}_X(U) : P_Z = P\}$$

for a given $P \in \mathbb{Z}[m]$.

Theorem 1.5. *For any $P \in \mathbb{Z}[m]$, the subfunctor $Hilb_X^P(U)$ is representable by a projective Noetherian scheme over k.*

Remark 1.10. In general, in order to solve a moduli problem in algebraic geometry, we first construct a functor which is expected to be representable by the moduli space we are looking for. Second, prove (using general categorical facts) that such a functor is indeed representable. The idea is to first consider general presheaves and then restrict to the subcategory of representable presheaves. In particular, we need a "practical" criteria for representability of functors. We will restrict to Algebraic Geometry.

Let $\mathcal{C} = \mathbf{Schemes}_k$ be the category of schemes over some commutative ring k. The Yoneda embedding

$$h : \mathbf{Schemes}_k \hookrightarrow \mathbf{Fun}\left(\mathbf{Schemes}_k^{op}, \mathbf{Set}\right)$$

is fully faithful and we can refine it using

$$\mathbf{CommAlg}_k \xrightarrow[\text{Spec}]{\sim} \mathbf{AffineSchemes}_k^{op} \hookrightarrow \mathbf{Schemes}_k^{op}$$

by restricting the Yoneda functor to commutative algebras to obtain

$$\bar{h} : \mathbf{Schemes}_k \hookrightarrow \mathbf{Fun}\left(\mathbf{Schemes}_k^{op}, \mathbf{Set}\right) \longrightarrow \mathbf{Fun}\left(\mathbf{CommAlg}_k, \mathbf{Set}\right).$$

Lemma 1.4. (Refined Yoneda Lemma) *The functor* $\bar{h} : \mathbf{Schemes}_k \hookrightarrow \mathbf{Fun}\left(\mathbf{CommAlg}_k, \mathbf{Set}\right)$ *is fully faithful. The corresponding essential image consists of scheme-functors.*

For a proof of this theorem, see Chapter 4, proposition 4-2 in [EH]. Note that the representable functors among these scheme functors are precisely affine schemes.

Problem. Characterize scheme functors among all functors from $\mathbf{CommAlg}_k$ to $\mathbf{Set}$.

Definition 1.18. A functor $F : \mathbf{CommAlg}_k \longrightarrow \mathbf{Set}$ is called a *sheaf* in the Zariski topology if for any $A \in \mathbf{CommAlg}_k$ and any open covering of $\operatorname{Spec}(A)$ by distinguished open affine sets $U_i = \operatorname{Spec}(A_{f_i})$ where A_{f_i} is the localization of A at some $f_i \in A$, the following sheaf axiom holds:

[SA] For any choice of $\alpha_i \in FA_{f_i}$ such that

$$\rho^{A_{f_i}}_{A_{f_i f_j}}(\alpha_i) = \rho^{A_{f_j}}_{A_{f_i f_j}}(\alpha_j)$$

for all i, j, then there exists a unique $\alpha \in FA$ such that $\rho^{A}_{A_{f_i}}(\alpha) = \alpha_i$, where $\rho^{A_{f_i}}_{A_{f_i f_j}} : FA_{f_i} \longrightarrow FA_{f_i f_j}$.

Theorem 1.6. *A functor* $F : \mathbf{CommAlg}_k \longrightarrow \mathbf{Set}$ *is a scheme functor if and only if*

1. *F is a sheaf in the Zariski topology.*

2. *There exists a k-algebra A_i and $\alpha_i \in FA_i$ such that*

$$\hat{\alpha}_i : h^{A_i} \Rightarrow F$$

satisfies the property: for all fields $K \supset k$, the image of $h^{A_i}(K)$ under $\hat{\alpha}_i$ covers FK.

Generalized Manifolds.

Let **Man** be the category of finite dimensional $\mathbf{C}^{\infty}$-smooth manifolds.

Definition 1.19. A *generalized manifold* is a functor (presheaf) $F : \mathbf{Man}^{op} \to \mathbf{Set}$ which satisfies the sheaf axiom, i.e. for any $M \in Ob(\mathbf{Man})$, for any open cover $\{U_i\}_{i\in I}$ of M, $M = \cup_{i\in I} U_i$, if for any $\alpha_i \in FU_i$ we have $\rho^{U_i}_{U_{ij}}(\alpha_i) = \rho^{U_j}_{U_{ij}}(\alpha_j)$, then there exists a unique $\alpha \in FM$ such that $\rho^{M}_{U_i}(\alpha) = \alpha_i$.

By Yoneda lemma,

$$\begin{array}{rcl} \mathbf{Man} & \hookrightarrow & \mathbf{GenMan} \subseteq \widehat{\mathbf{Man}} \\ M & \mapsto & h_M = \mathrm{Map}(-, M) : \quad N \quad \mapsto \quad \mathrm{Map}(N, M) \end{array}$$

Question. What are generalized manifolds that are not honest manifolds?

For $M \in Ob(\mathbf{Man})$, $p \geq 1$, consider $\Omega^p(M)$ the space of differential forms of degree p.

$$\Omega^p(M) = \left\{ \omega = \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p}(x)\, dx_{i_1} \wedge \cdots dx_{i_p} \right\}$$

The map

$$\begin{array}{rcl} M & \mapsto & \Omega^p(M) \\ (f : M \to N) & \mapsto & (f^* : \Omega^p(N) \to \Omega^p(M)) \end{array}$$

is functorial in M. This makes $\Omega^p : \mathbf{Man}^{op} \to \mathbf{Set}$ a contravariant functor.

Theorem 1.7. *For $p \geq 1$, Ω^p is a generalized manifold.*

Point. We can extend differential calculus to generalized manifolds.

Definition 1.20. Given a generalized manifold $F : \mathbf{Man}^{op} \to \mathbf{Set}$, the *set of differential forms on F of degree p* is

$$\Omega^p(F) := \mathrm{Hom}_{\widehat{\mathbf{Man}}}(F, \Omega^p)$$

For example, given a generalized manifold $F : \mathbf{Man}^{op} \to \mathbf{Set}$, we can define $\Omega^p(F)$ in such way

$\Omega^p(h_M) = \Omega^p(M)$. Recall Yoneda lemma tells us that

$$\mathrm{Hom}_{\widehat{\mathbf{Man}}}(h_M, F) \cong FM.$$

In particular, take $F = \Omega^p$, $\mathrm{Hom}_{\widehat{\mathbf{Man}}}(h_M, \Omega^p) \cong \Omega^p(M), \forall M \in \mathbf{Man}$.

Recall that differential forms on M define de Rham complex

$$\Omega^\bullet(M) \coloneqq \left[\Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \cdots \xrightarrow{d} \Omega^n(M) \to \cdots\right]$$

where d is the de Rham differential. This operator id functorial.

$$\begin{aligned} M &\mapsto \Omega^\bullet(M) \\ (f : M \to N) &\mapsto (f^* : \Omega^\bullet(N) \to \Omega^\bullet(M)) \end{aligned}$$

The de Rham cohomology is defined as $H^*_{DR}(M) \coloneqq H^*(\Omega^\bullet(M), d)$.

This extends to a definition of de Rham complex $(\Omega^\bullet(F), d)$, and $H^*_{DR}(F) = H^*(\Omega^\bullet(F), d)$.

Take $F = \Omega^1 : \mathbf{Man}^{op} \to \mathbf{Set}$, complete the de Rham complex, $\Omega^p(\Omega^1) = \mathrm{Hom}_{\widehat{\mathbf{Man}}}(\Omega^1, \Omega^p)$ is the space of all natural constructions of a p-form on a manifold from 1-form.

Theorem 1.8. *The de Rham complex of Ω^1 is isomorphic to the following complex,*

$$\Omega^*(\Omega^1) \cong \left[\mathbf{R} \xrightarrow{0} \mathbf{R} \xrightarrow{Id} \mathbf{R} \xrightarrow{0} \mathbf{R} \xrightarrow{Id} \cdots\right]$$

and

$$H^p_{DR}(\Omega^1) = \begin{cases} \mathbf{R}, & p = 0, \\ 0 & o.w. \end{cases}$$

1.7 Adjoint Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Fix $Y \in Ob(\mathcal{D})$ and define

$$\begin{aligned} \widetilde{F}_Y : \mathcal{C}^{op} &\to \mathbf{Set} \\ T &\mapsto \mathrm{Hom}_{\mathcal{D}}(FT, Y) \end{aligned}$$

or equivalently, $\widetilde{F}_Y = h_Y \circ F$.

Theorem 1.9. *Assume that $\widetilde{F}_Y$ is representable for each $Y \in Ob(\mathcal{D})$, with a representable $X = X_Y \in Ob(\mathcal{C})$, $\widetilde{F}_Y \cong h_X$. Then the assignments $Y \mapsto X_Y$ extends to a functor $G : \mathcal{D} \to \mathcal{C}$ such that there is an isomorphism of bifunctors $\mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$,*

$$\eta : \quad Hom_{\mathcal{D}}(F(-), -) \quad \longrightarrow \quad Hom_{\mathcal{C}}(-, G(-))$$

This G is unique up to a unique isomorphism.

Remark 1.11. The theorem can be restated as follows, $F : \mathcal{C} \to \mathcal{D}$ defines a functor

$$\begin{aligned} F^* : \quad \widehat{\mathcal{D}} \quad &\to \quad \widehat{\mathcal{C}} \\ H \quad &\mapsto \quad H \circ F \end{aligned}$$

Furthermore, we can consider $h_{\mathcal{D}} : \mathcal{D} \to \widehat{\mathcal{D}}$ and $F^* \circ h_{\mathcal{D}} : \mathcal{D} \hookrightarrow \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$. On the other hand, given $G : \mathcal{D} \to \mathcal{C}$, we can consider $h_{\mathcal{C}} \circ G : \mathcal{D} \to \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$.

This theorem is equivalent to say that there exists a unique $G : \mathcal{D} \to \mathcal{C}$ with isomorphism $F^* \circ h_{\mathcal{D}} = h_{\mathcal{C}} \circ G$. Here uniqueness means that if there are two functors $G_1, G_2 : \mathcal{D} \to \mathcal{C}$ with $\varphi_i : F^* \circ h_{\mathcal{D}} \overset{\sim}{\Rightarrow} h_{\mathcal{C}} \circ G_i, i = 1, 2$, then there exists a unique $f : G_1 \Rightarrow G_2$ such that $\varphi_2 = (h_{\mathcal{C}} \circ f)\, \varphi_1$.

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "h_{\mathcal{C}}"] & \\
\mathcal{D} \arrow[ur, "G_1"] \arrow[r, hook, "h_{\mathcal{D}}"] \arrow[dr, "G_2"'] & \widehat{\mathcal{D}} \arrow[u, Rightarrow, "\varphi_1"'] \arrow[d, Rightarrow, "\varphi_2"] \arrow[r, "F^*"] & \mathcal{C} \\
 & \mathcal{C} \arrow[ur, "F^*"'] &
\end{tikzcd}$$

Proof. (See details in lecture notes [HA])

For $Y \in Ob(\mathcal{D})$, denote the isomorphism by $\psi : h_X \overset{\sim}{\Rightarrow} \widetilde{F}_Y$, i.e.

$$\psi = \{\psi_T : \mathrm{Hom}_{\mathcal{C}}(T, X_Y) \to \mathrm{Hom}_{\mathcal{D}}(FT, Y)\}_{T \in Ob(\mathcal{C})}$$

Take $T = X_Y$, define

$$\sigma_{X_Y} := \psi_{X_Y}(\mathrm{Id}_{X_Y}) : FX_Y \to Y.$$

Now we define $G : \mathcal{D} \to \mathcal{C}$ by

- Objects: $G : Y \mapsto X_Y$.
- Morphisms: Given $f : Y \to \widetilde{Y}$, consider the bijection of sets

$$\begin{array}{rccc} \tilde{\psi}: & \mathrm{Hom}_{\mathcal{C}}\left(-, X_{\tilde{Y}}\right) & \to & \mathrm{Hom}_{\mathcal{D}}\left(F\left(-\right), \tilde{Y}\right) \\ & \| & & \| \\ & h_{X_{\tilde{Y}}} & & \widetilde{F}_{\tilde{Y}} \end{array}$$

  and

$$f \circ \sigma_{GY} : FX_Y \to Y \to \tilde{Y},$$

  define

$$G\left(f\right) \coloneqq \tilde{\psi}_{GY}^{-1}\left(f \circ \sigma_{GY}\right) : X_Y \to X_{\tilde{Y}}.$$

It is straightforward to show that $G : \mathcal{D} \to \mathcal{C}$ is a functor and satisfies the condition of the theorem.

□

Definition 1.21. The functor $G : \mathcal{D} \to \mathcal{C}$ given by the theorem is called *right adjoint functor* of F, we write

$$F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$$

or

$$\begin{tikzcd} \mathcal{C} \arrow[d, bend right, "F"'] \\ \mathcal{D} \arrow[u, bend right, "G"'] \end{tikzcd}$$

The adjoint pair (F, G) comes together with isomorphisms

$$\eta_{X,Y} : \mathrm{Hom}_{\mathcal{D}}\left(FX, Y\right) \to \mathrm{Hom}_{\mathcal{C}}\left(X, GY\right), \qquad \forall X \in Ob\left(\mathcal{C}\right), Y \in Ob\left(\mathcal{D}\right) \tag{6}$$

which are natural in X, Y.

Adjunction Morphisms

Let (F, G, η) be an adjoint pair of functors, then

1. Take $Y = FX$ in 6 and apply to Id_{FX}, we get

$$\sigma_X \coloneqq \eta_{X,FX}\left(\mathrm{Id}_{FX}\right) : X \to GFX, \tag{7}$$

which gives a morphism of functors

$$\sigma : \mathrm{Id}_{\mathcal{C}} \to GF$$

called the *unit* of adjunction.

2. Take $X = GY$ in 6 and apply to Id_{GY}, we get

$$\varepsilon_Y \coloneqq \eta^{-1}_{GY,Y}\left(\mathrm{Id}_{GY}\right) : FGY \to Y, \tag{8}$$

which gives a morphism of functors

$$\varepsilon : FG \to \mathrm{Id}_{\mathcal{D}}$$

called the *counit* of adjunction.

We will write $(F, G, \sigma, \varepsilon)$.

Using convolution of functors and morphisms of functors, define $F\circ\sigma : F \to FGF$, $\{\sigma_X : X \to GFX\}_{X\in Ob(\mathcal{C})}$ gives $\{(F\circ\sigma)_X : FX \to FGFX\}_{X\in Ob(\mathcal{C})}$, and $G\circ\varepsilon : GFG \to G$, $\{\varepsilon_Y : FGY \to Y\}_{Y\in Ob(\mathcal{D})}$ gives $\{(G\circ\varepsilon)_Y : GFGY \to GY\}_{Y\in Ob(\mathcal{D})}$.

Similarly, we can define $\sigma\circ G : G \to GFG$ and $\varepsilon\circ F : FGF \to F$.

Observation. η can be recovered from σ and ε as follows.

By functoriality of $\eta_{X,-} : \mathrm{Hom}_{\mathcal{D}}(FX, -) \to \mathrm{Hom}_{\mathcal{C}}(X, G(-))$, for any $\varphi : FX \to Y$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(FX, FX) \arrow[r, "\eta_{X,FX}"] \arrow[d, "\varphi_*"'] & \mathrm{Hom}_{\mathcal{C}}(X, GFX) \arrow[d, "(G\varphi)_*"] \\
\mathrm{Hom}_{\mathcal{D}}(FX, Y) \arrow[r, "\eta_{X,Y}"] & \mathrm{Hom}_{\mathcal{C}}(X, GY)
\end{tikzcd}$$

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \eta_{X,FX}(f) \arrow[d, mapsto] \\ \varphi \circ f \arrow[r, mapsto] & \eta_{X,Y}(\varphi \circ f) = G\varphi \circ \eta_{X,FX}(f) \end{tikzcd}$$

Apply to $f = \mathrm{Id}_{FX}$, we have

$$\eta_{X,Y}(\varphi) = G\varphi \circ \eta_{X,FX}(\mathrm{Id}_{FX}) = G\varphi \circ \sigma_X. \tag{9}$$

Dually for any $\psi : X \to GY$ we have

$$\eta_{X,Y}^{-1}(\psi) = \varepsilon_Y \circ F(\psi). \tag{10}$$

Thus giving (F, G, η) is equivalent to giving $(F, G, \sigma, \varepsilon)$.

Lemma 1.5. *For any adjoint pair (F, G) the following identities hold.*

$$Id_F = (\varepsilon \circ F)(F \circ \sigma) : F \xrightarrow{F\circ\sigma} FGF \xrightarrow{\varepsilon\circ F} F \tag{11}$$

$$Id_G = (G \circ \varepsilon)(\sigma \circ G) : G \xrightarrow{\sigma\circ G} GFG \xrightarrow{G\circ\varepsilon} G \tag{12}$$

Proof. Straightforward. □

Proposition 1.2. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors, Then $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ if and only if there exists $\sigma : Id_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to Id_{\mathcal{D}}$ satisfying 11 and 12.*

Proof. " $\Longrightarrow$ " Assume $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ are adjoint with η as in 6. Define σ and ε as described in the equations 7 and 8. Using the relations 9 and 10, for any $\varphi : FX \to Y$,

$$\varphi = \eta_{X,Y}^{-1}(\eta_{X,Y}(\varphi)) = \eta_{X,Y}^{-1}(G\varphi \circ \sigma_X) = \varepsilon_Y \circ F(G\varphi \circ \sigma_X) = \varepsilon_Y \circ FG\varphi \circ F\sigma_X$$

Take $Y = FX$, $\varphi = \mathrm{Id}_{FX}$, then $\mathrm{Id}_{FX} = \varepsilon_{FX} \circ FG(\mathrm{Id}_{FX}) \circ F\sigma_X = \varepsilon_{FX} \circ F\sigma_X$, which is the same as the equality 11. Similarly we have the equality 12.

" $\Longleftarrow$ " Assume $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are two functors and there exists $\sigma : \mathrm{Id}_{\mathcal{C}} \to GF$ and

$\varepsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ satisfying the equalities 11 and 12. Define for any $X \in Ob(\mathcal{C})$ and $Y \in Ob(\mathcal{D})$,

$$\begin{array}{rccc} \eta_{X,Y}: & \mathrm{Hom}_{\mathcal{D}}(FX, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X, GY) \\ & \varphi & \longmapsto & G\varphi \circ \sigma_X \end{array}$$

and

$$\begin{array}{rccc} \eta'_{X,Y}: & \mathrm{Hom}_{\mathcal{C}}(X, GY) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(FX, Y) \\ & \psi & \longmapsto & \varepsilon_Y \circ F\psi \end{array}$$

We will prove that $\eta' = \eta^{-1}$.

Consider the diagram

$$\begin{tikzcd} FX \arrow[r, "F\sigma_X"] \arrow[dr, "\mathrm{Id}_{FX}"'] \arrow[dr, phantom, "I", shift left=3] & FGFX \arrow[r, "FG\varphi"] \arrow[d, "\varepsilon_{FX}"] \arrow[dr, phantom, "II"] & FGY \arrow[d, "\varepsilon_Y"] \\ & FX \arrow[r, "\varphi"] & Y \end{tikzcd}$$

Note (I) commutes by the equality 11 and (II) commutes by the fact that ε is a morphism of functors. So we have

$$\varphi = \varphi \circ \mathrm{Id}_{FX} = \varepsilon_Y \circ FG\varphi \circ F\sigma_X = \varepsilon_Y \circ F(G\varphi \circ \sigma_X) = \eta'_{X,Y} \circ \eta_{X,Y}(\varphi).$$

So $\eta'_{X,Y}\eta_{X,Y} = \mathrm{Id}$. Similarly, $\eta_{X,Y}\eta'_{X,Y} = \mathrm{Id}$. Therefore $\eta'_{X,Y} = \eta^{-1}_{X,Y}, \forall X \in Ob(\mathcal{C}), Y \in Ob(\mathcal{D})$. □

Examples/ Applications

Corollary 1.3. *If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories with a quasi-inverse $G : \mathcal{D} \to \mathcal{C}$, then G is both right and left adjoint of F.*

$$F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$$

$$G : \mathcal{D} \rightleftharpoons \mathcal{C} : F$$

Proof. $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories with a quasi-inverse $G : \mathcal{D} \to \mathcal{C}$ if and only if there exists two natural isomorphisms $\alpha : GF \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}}$ and $\beta : FG \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$.

Take $\sigma = \alpha^{-1} : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\varepsilon = \beta : FG \to \mathrm{Id}_{\mathcal{D}}$, satisfying 11 and 12, so $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$.

Take $\sigma = \beta^{-1} : \mathrm{Id}_{\mathcal{D}} \to FG$ and $\varepsilon = \alpha : GF \to \mathrm{Id}_{\mathcal{C}}$, satisfying 11 and 12, so $G : \mathcal{D} \rightleftharpoons \mathcal{C} : F$. □

Corollary 1.4. *Let $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ be adjoint functors. For any category $\mathcal{A}$, there are adjunctions* *of the restriction functors*

$$G^* : \mathbf{Fun}(\mathcal{C}, \mathcal{A}) \rightleftharpoons \mathbf{Fun}(\mathcal{D}, \mathcal{A}) : F^*$$

given by $\left(\mathcal{C} \xrightarrow{R} \mathcal{A}\right) \longmapsto \left(\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{R} \mathcal{A}\right) =: G^*(R)$ *and* $F^*(L) := \left(\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{L} \mathcal{A}\right) \longleftarrow \left(\mathcal{D} \xrightarrow{L} \mathcal{A}\right)$.

Proof. Let $\sigma : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ be the adjunction morphisms for the adjoint pair (F, G). Given $\left(\mathcal{C} \xrightarrow{R} \mathcal{A}\right) \in \mathrm{Ob}(\mathbf{Fun}(\mathcal{C}, \mathcal{A}))$, consider $R\sigma : R \to RGF$ and vary R in $\mathbf{Fun}(\mathcal{C}, \mathcal{A})$ to obtain

$$\sigma^* : \mathrm{Id}_{\mathbf{Fun}(\mathcal{C},\mathcal{A})} \longrightarrow F^*G^* \left(= (GF)^*\right) \qquad \text{and} \qquad \varepsilon^* : G^*F^* \longrightarrow \mathrm{Id}_{\mathbf{Fun}(\mathcal{D},\mathcal{A})}.$$

Since σ and ε satisfy 11 and 12, so do σ^* and ε^*. In particular, (G^*, F^*) is an adjoint pair. □

Theorem 1.10. *(Freyd) A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Proof. It is clear by definition that an equivalence of categories is fully faithful and essentially surjective. Conversely, for any $Y \in \mathrm{Ob}(\mathcal{D})$, consider the functor

$$\mathrm{Hom}_{\mathcal{D}}(F(-), Y) : \mathcal{C}^{op} \longrightarrow \mathbf{Set}.$$

Since F is essentially surjective, there exists $X \in \mathrm{Ob}(\mathcal{C})$ together with isomorphism $FX \cong Y$ so that

$$\mathrm{Hom}_{\mathcal{D}}(F(-), Y) \cong \mathrm{Hom}_{\mathcal{D}}(F(-), FX) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, X) = h_X.$$

In particular, $\mathrm{Hom}_{\mathcal{D}}(F(-), Y)$ is representable for all $Y \in \mathrm{Ob}(\mathcal{D})$ and hence has a right adjoint $G : \mathcal{D} \to \mathcal{C}$ together with morphisms $\sigma : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to \mathrm{Id}_{\mathcal{D}}$. We now show that σ and ε are isomorphisms.

Fix $X' \in \mathrm{Ob}(\mathcal{C})$ and for all $X \in \mathrm{Ob}(\mathcal{C})$, apply the functor $\mathrm{Hom}_{\mathcal{C}}\left(X', -\right)$ to the morphism $\sigma_X : X \to GFX$ to get $\mathrm{Hom}_{\mathcal{C}}\left(X', X\right) \to \mathrm{Hom}_{\mathcal{C}}\left(X', GFX\right)$. Notice that the resulting morphism factors as

$$\begin{tikzcd}
h_X\left(X'\right) = \mathrm{Hom}_{\mathcal{C}}\left(X', X\right) \arrow[rr] \arrow[dr, "F", "\cong"'] & & \mathrm{Hom}_{\mathcal{C}}\left(X', GFX\right) = h_{GFX}\left(X'\right) \\
& \mathrm{Hom}_{\mathcal{C}}\left(FX', FX\right) \arrow[ur, "\eta_{X',FX}", "\cong"'] &
\end{tikzcd}$$

so is a bijection of sets for every $X, X^{'} \in \mathrm{Ob}(\mathcal{C})$ and hence σ_X induces an isomorphism of functors $h_X \cong h_{GFX}$ represented by X and GFX. The Yoneda lemma then implies $X \cong GFX$. Dually, start with $\varepsilon_Y : FGY \to Y$ for every $Y \in \mathrm{Ob}(\mathcal{D})$ and apply the functor $\mathrm{Hom}_{\mathcal{D}}(FX, -)$ to ε_Y to obtain

$$\phi : \mathrm{Hom}_{\mathcal{D}}(FX, FGY) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(FX, Y)$$

for all $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$. Composing ϕ with the isomorphism resulting from F being fully faithful, we obtain the commuting diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X, GY) \arrow[rr, "\eta^{-1}_{X,Y}", "\cong"'] \arrow[dr, "F", "\cong"'] & & \mathrm{Hom}_{\mathcal{D}}(FX, Y) \\ & \mathrm{Hom}_{\mathcal{D}}(FX, FGY) \arrow[ur, "\phi"] & \end{tikzcd}$$

and ϕ is an isomorphism. By essential surjectivity of F, for all $Y^{'} \in \mathrm{Ob}(\mathcal{D})$, there exists $X \in \mathrm{Ob}(\mathcal{C})$ so that $Y^{'} \cong FX$. In particular, the isomorphism ϕ induces an isomorphism representable functors

$$h_{FGY}\left(Y^{'}\right) = \mathrm{Hom}_{\mathcal{D}}\left(Y^{'}, FGY\right) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}\left(Y^{'}, Y\right) = h_Y\left(Y^{'}\right).$$

The Yoneda lemma then implies that $FGY \cong Y$. □

Corollary 1.5. *Let $(F, G, \sigma, \varepsilon)$ be an adjunction. Then*

1. *G is fully faithful if and only if $\varepsilon : GF \longrightarrow Id_{\mathcal{D}}$ is an isomorphism.*
2. *F is fully faithful if and only if $\sigma : Id_{\mathcal{C}} \longrightarrow FG$ is an isomorphism.*
3. *Both F and G fully faithful if and only if F is an equivalence of categories if and only if G is an equivalence of categories.*

Lemma 1.6. *Let $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ and $L : \mathcal{D} \rightleftharpoons \mathcal{E} : R$ be adjunctions. Then $LF : \mathcal{C} \rightleftharpoons \mathcal{E} : GR$ is an adjunction.*

Proof. For every $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{E})$, $\mathrm{Hom}_{\mathcal{E}}(LFX, Y) \cong \mathrm{Hom}_{\mathcal{D}}(FX, RY) \cong \mathrm{Hom}_{\mathcal{C}}(X, GRY)$. □

Traces in Categories. Given a triple (E, F, G) of adjoint functors

$$\begin{tikzcd} \mathcal{C} \arrow[d, "F"] \\ \mathcal{D} \arrow[u, bend left=60, "E"] \arrow[u, bend right=60, "G"'] \end{tikzcd}$$

and a morphism of functors $\nu : G \Rightarrow E$, we have a natural *trace map* for all $X, Y \in \mathrm{Ob}\,(\mathcal{C})$

$$\mathrm{tr} : \mathrm{Hom}_{\mathcal{D}}\,(FX, FY) \longrightarrow \mathrm{Hom}_{\mathcal{C}}\,(X, Y)$$

defined for any $\varphi : FX \to FY$ as the following composition:

$$X \xrightarrow{\sigma_X} GFX \xrightarrow{\nu_{FX}} EFX \xrightarrow{E(\varphi)} EFY \xrightarrow{\varepsilon_Y} Y.$$

Taking $X = Y$, define

$$\mathrm{tr}\,(F) = \left\{\mathrm{tr}\,(F)_X : \mathrm{End}_{\mathcal{D}}\,(FX) \longrightarrow \mathrm{End}_C\,(X)\right\}_{X \in \mathrm{Ob}(\mathcal{C})}.$$

The map $\mathrm{tr}\,(F)$ is a morphism of functors $\mathrm{tr}\,(F) : \mathrm{End}\,(F) \Rightarrow \mathrm{End}\,(\mathrm{Id}_{\mathcal{C}})$ where $\mathrm{End}\,(F) := \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C}.\mathcal{D})}\,(F, F)$ and $\mathrm{End}\,(\mathrm{Id}_{\mathcal{C}})$, defined similarly, is the center of the category $\mathcal{C}$. This trace map is called the *Bernstein trace map.*

We compute this trace map more explicitly for $\mathcal{C} = \mathcal{D} = \mathbf{Vect}_k$, the category of vector spaces over the field k. Fix a finite dimensional vector space $V \in \mathbf{Vect}_k$ and consider

$$\begin{array}{rccc} F_V := - \otimes_k V : & \mathbf{Vect}_k & \longrightarrow & \mathbf{Vect}_k \\ & W & \longmapsto & W \otimes_k V. \end{array}$$

The functor F_V has a right adjoint:

$$\mathrm{Hom}_k\,(F_V W, M) = \mathrm{Hom}_k\,(W \otimes_k V, M) \overset{(1)}{\cong} \mathrm{Hom}_k\,(W, \mathrm{Hom}_k\,(V, M)) \overset{(2)}{\cong} \mathrm{Hom}_k\,(W, M \otimes_k V^*)$$

where we use the notation Hom_k to mean $\mathrm{Hom}_{\mathbf{Vect}_k}$. The canonical isomorphism (1) is the usual tensor-hom adjunction which is given explicitly by

$$\begin{pmatrix} W \otimes_k V & \overset{\varphi}{\to} & M \\ w \otimes v & \mapsto & \varphi(w,v) \end{pmatrix} \longmapsto \begin{pmatrix} W & \to & \mathrm{Hom}_k(V,M) \\ w & \mapsto & (v \mapsto \varphi(w,v)) \end{pmatrix}.$$

The isomorphism (2) follows from the isomorphism $M \otimes_k V^* \cong \mathrm{Hom}_k(V,M)$ given explicitly by $m \otimes v^* \mapsto (\phi : v \mapsto v^*(v)\, m)$. Define $G := - \otimes_k V^*$, then G is right adjoint to F_V. Moreover, it is also a left adjoint because $V \cong V^{**}$ and applying the above construction to V^* gives the desired left adjoint

$$\begin{tikzcd} \mathbf{Vect}_k \arrow[d, "-\otimes_k V"'] \arrow[d, bend right=60, leftarrow, "-\otimes_k V^*"'] \arrow[d, bend left=60, leftarrow, "-\otimes_k V^*."] \\ \mathbf{Vect}_k \end{tikzcd}$$

For $V \in \mathbf{Vect}_k$ define natural maps

$$\begin{array}{rccc} \iota : & k & \longrightarrow & \mathrm{End}_k(V) \\ & 1 & \longmapsto & \mathrm{Id}_V \end{array} \qquad \begin{array}{rccc} \rho : & V \otimes_k V^* & \longrightarrow & k \\ & v \otimes w^* & \longmapsto & w^*(v) \end{array} \qquad \begin{array}{rccc} \mu : & V \otimes_k V^* & \longrightarrow & \mathrm{End}_k(V) \\ & v \otimes w^* & \longmapsto & (x \mapsto w^*(x)\, v) \end{array}.$$

If V is finite dimensional, then μ is an isomorphism and we denote its inverse by $\nu = \mu^{-1}$. We can compute the adjunction maps precisely:

The unit $\sigma : \mathrm{Id}_{\mathbf{Vect}_k} \to GF$ is defined for all $M \in \mathbf{Vect}_k$ as

$$\begin{array}{rl} \sigma_M : & M \longrightarrow M \otimes \mathrm{End}_k(V) \xrightarrow{\cong} M \otimes_k (V \otimes_k V^*) \xrightarrow{\qquad\cong\qquad} (M \otimes_k V) \otimes_k V^* \\ & m \longmapsto (\mathrm{Id}_M \otimes \nu)(m \otimes \mathrm{Id}_V) = (\mathrm{Id}_M \otimes (\nu \circ \iota))(m) \end{array}$$

The counit $\varepsilon : EF \to \mathrm{Id}_{\mathbf{Vect}_k}$ is defined for every $M \in \mathbf{Vect}_k$ by

$$\begin{array}{rl} \varepsilon_M : & (M \otimes_k V) \otimes_k V^* \xrightarrow{\cong} M \otimes_k (V \otimes V^*) \longrightarrow M \\ & (m \otimes v) \otimes w^* \longmapsto w^*(v)\, m \end{array}$$

which is precisely $\varepsilon_M = \mathrm{Id}_M \otimes \rho$. The trace map is given by $\mathrm{tr}(F_V) : \mathrm{End}(- \otimes_k V) \to \mathrm{End}(\mathrm{Id}_{\mathbf{Vect}_k})$

with

$$\operatorname{tr}(F_V)_M : \operatorname{End}(M \otimes_k V) \to \operatorname{End}_k(M)$$

for every $M \in \mathbf{Vect}_k$. In particular, every $\varphi : M \otimes_k V \to M \otimes_k V$ is mapped to the composition

$$\operatorname{tr}_M(F_V)(\varphi) : M \xrightarrow{\mathrm{Id}_M \otimes \iota} M \otimes \operatorname{End}_k(V) \xrightarrow{\mathrm{Id}_M \otimes \nu} M \otimes_k (V \otimes_k V^*) \xrightarrow{\varphi \otimes \mathrm{Id}_{V^*}} M \otimes_k V \otimes_k V^* \xrightarrow{\mathrm{Id}_M \otimes p} M \otimes_k k \cong M$$

which can be described explicitly by a choice of dual bases for V and V^*. Suppose V is n-dimensional and let $\{v_j\}_{j=1}^n$ and $\left\{v_j^*\right\}_{j=1}^n$ be dual bases, that is, $< v_j^*, v_j >= \delta_{ij}$. Any $\varphi : M \otimes_k V \to M \otimes_k V$ can be written with respect to this basis as

$$m \otimes v_j \longmapsto \varphi(m \otimes v_j) = \sum_{i=1}^n \varphi_{ij}(m) \otimes v_i$$

where $\|\varphi_{ij}\|_{i,j=1}^n \in \mathbb{M}_n(\operatorname{End}(M))$. This allows us to describe the map $\nu : \operatorname{End}_k(V) \to V \otimes_k V^*$ explicitly as

$$\nu(\varphi) = \sum_{i=1}^n \varphi(v_i) \otimes v_i^*.$$

For $m \in M$,

$$\operatorname{tr}(F_V)_M(\varphi) : \ m \longmapsto m \otimes \mathrm{Id}_V \longmapsto \textstyle\sum_{j=1}^n m \otimes v_j \otimes v_j^* \longmapsto \sum_{i,j=1}^n \varphi_{ij}(m) \otimes v_i \otimes v_j^* \longmapsto \sum_{i,j=1}^n \varphi_{ij}(m)\, \delta_{ij} \ .$$

Thus,

$$\operatorname{tr}(F_V)_M(\varphi)(m) = \sum_{j=1}^n \varphi_{jj}(m) \in \operatorname{End}_k(M)$$

and is known as *relative trace.* In the special case when $M = k$, $\operatorname{End}_k(k) \cong k$ and

$$\begin{array}{rccc} \operatorname{tr}(F_V)_k : & \operatorname{End}_k(V) & \longrightarrow & k \\ & \varphi & \longmapsto & \sum_{j=1}^n \varphi_{jj} \end{array}$$

which is the usual trace of matrices.

Representation Theory.

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. A practical example to keep in mind is $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

Let $\mathfrak{g}-\mathbf{Mod}$ be the category of representations of $\mathfrak{g}$. We can identify $\mathfrak{g}-\mathbf{Mod}$ with the category of left modules over an associative algebra $U_{\mathfrak{g}}$, the *universal enveloping algebra.* One way to define the universal enveloping algebra is via the motto:

Left-adjoint functors to forgetful functors solve universal problems.

In this example, there is a forgetful functor

$$\mathbf{LieAlg}_k \longleftarrow \mathbf{Alg}_k : F$$

from the category of algebras over k to the category of Lie algebras over k that induces on an algebra A the obvious Lie algebra structure. This functor has a left adjoint, the universal enveloping functor,

$$U : \mathbf{LieAlg}_k \longrightarrow \mathbf{Alg}_k$$

that maps any Lie algebra $\mathfrak{a}$ to its universal enveloping algebra $U_{\mathfrak{a}}$.

Exercise. Show that for any Lie algebra $\mathfrak{a}$, its universal enveloping algebra $U_{\mathfrak{a}}$ has the following form

$$U_{\mathfrak{a}} = T\mathfrak{a} / \langle x \otimes y - y \otimes x - [x, y]_{\mathfrak{a}} : x, y \in \mathfrak{a} \rangle$$

where $T\mathfrak{a}$ is the tensor algebra of $\mathfrak{a}$.

For a Lie algebra $\mathfrak{g}$, there is an isomorphism of categories $\mathfrak{g}-\mathbf{Mod} \cong U_{\mathfrak{g}}-\mathbf{Mod}$ resulting from the (U, F) adjunction which gives the isomorphism

$$\mathrm{Hom}_{\mathbf{LieAlg}_k}(\mathfrak{g}, F(\mathrm{End}(V))) \cong \mathrm{Hom}_{\mathbf{Alg}_k}(U_{\mathfrak{g}}, \mathrm{End}(V)).$$

Take a finite dimensional representation of $\mathfrak{g}$, say V, and consider

$$\begin{array}{rccc} F_V = - \otimes_k V : & \mathfrak{g}-\mathbf{Mod} & \longrightarrow & \mathfrak{g}-\mathbf{Mod} \\ & W & \longmapsto & W \otimes_k V \end{array}$$

where $W \otimes_k V$ is again a $\mathfrak{g}$ representation via the formula $\xi(w \otimes v) = \xi w \otimes v + w \otimes \xi v$. Consider the trace map

$$\operatorname{tr}(F_V) : \operatorname{End}(F_V) \to \operatorname{End}(\operatorname{Id}_{\mathfrak{g}-\mathbf{Mod}}).$$

The target of this trace map is well understood, in particular,

$$\operatorname{End}(\operatorname{Id}_{\mathfrak{g}-\mathbf{Mod}}) \cong \operatorname{End}\left(\operatorname{Id}_{U_{\mathfrak{g}}-\mathbf{Mod}}\right) =: \mathcal{Z}(\mathcal{U}_{\mathfrak{g}}-\mathbf{Mod}).$$

In fact, we proved in Theorem 1.31 that there is a natural isomorphism $Z(U_{\mathfrak{g}}-\mathbf{Mod}) \cong Z(U_{\mathfrak{g}}) \subset U_{\mathfrak{g}}$.

Theorem 1.11. *(Chevalley) For any semisimple complex Lie algebra* $\mathfrak{g}$*,*

$$U_{\mathfrak{g}} \cong Sym(\mathfrak{h})^W \cong \mathbb{C}[\mathfrak{h}^*]^W,$$

where $\mathfrak{h}$ *is the Cartan subalgebra of* $\mathfrak{g}$ *and* W *is the Weyl group.*

Proof. The Lie algebra $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$ so that $U_{\mathfrak{g}} = U_{\mathfrak{h}} \oplus (\mathfrak{n}U_{\mathfrak{g}} \oplus U_{\mathfrak{g}}\mathfrak{n}^-)$. Hence the map $\psi : U_{\mathfrak{g}} \twoheadrightarrow U_{\mathfrak{h}} = \operatorname{Sym}(\mathfrak{h})$ restricts to an isomorphism

$$\psi|_{\mathcal{Z}(\mathfrak{g})} : \mathcal{Z}(\mathfrak{g}) \longrightarrow \operatorname{Sym}(\mathfrak{h})^W$$

called the Chevalley isomorphism. □

Note that $\operatorname{End}(\operatorname{Id}_{\mathfrak{g}}) \cong \mathcal{Z}(\mathcal{U}_{\mathfrak{g}}) \cong \mathbb{C}[\mathfrak{h}^*]^W$ so there is a natural map

$$\begin{tikzcd}
\operatorname{End}(\operatorname{Id}_{\mathfrak{g}}) \arrow[r, "i"] \arrow[d, "\cong"] & \operatorname{End}(F_V) \arrow[r, "\operatorname{tr}(F_V)"] & \operatorname{End}(\operatorname{Id}_{\mathfrak{g}}) \arrow[d, "\cong"'] \\
\mathbb{C}[\mathfrak{h}^*]^W \arrow[rr, "\operatorname{tr}(F_V)\circ i"] & & \mathbb{C}[\mathfrak{h}^*]^W
\end{tikzcd}$$

with i given by $Z(U_{\mathfrak{g}}) \ni z \mapsto [(\varphi : F_V \to F_V) \mapsto (z.\varphi : F_V \to F_V)]$.

It would be interesting to also be able to describe explicitly the domain $\operatorname{End}(F_V)$ of the trace map.

The representation functor.

Fix $n \geq 1$, consider the $(n \times n)$-matrix functor

$$\begin{array}{rccc} \mathbb{M}_n : & \mathbf{Alg}_k & \to & \mathbf{Alg}_k \\ & B & \mapsto & \mathbb{M}_n(B) \end{array}$$

where k is a field, $\mathbf{Alg}_k$ is the category of associative unital k-algebra, and $\mathbb{M}_n(B)$ is the algebra of $(n \times n)$-matrices with entries in B.

Theorem 1.12. *This functor has left adjoint*

$$\begin{array}{rccc} \sqrt[n]{-} : & \mathbf{Alg}_k & \longrightarrow & \mathbf{Alg}_k \\ & A & \longmapsto & \sqrt[n]{A} \end{array}$$

the n-th noncommutative representation functor.

Corollary 1.6. *The classical representation scheme for a fixed associative algebra A*

$$\begin{array}{rccc} Rep_n(A) : & \mathbf{CommAlg}_k & \longrightarrow & \mathbf{Set} \\ & B & \longmapsto & Hom_{\mathbf{Alg}_k}(A, \mathbb{M}_n(B)) \end{array}$$

is represented by the commutative algebra $A_n = \left(\sqrt[n]{A}\right)_{ab} \coloneqq \sqrt[n]{A}/ \ll \sqrt[n]{A}, \sqrt[n]{A} \gg$.

Proof. Note the inclusion functor $i : \mathbf{CommAlg}_k \to \mathbf{Alg}_k$ has left adjoint

$$\begin{array}{rccc} (-)_{ab} : & \mathbf{Alg}_k & \longrightarrow & \mathbf{CommAlg}_k \\ & A & \longmapsto & A/ \ll [A, A] \gg \end{array}$$

the abelianization of A. Indeed, $\mathrm{Hom}_{\mathbf{Alg}_k}(A, \mathbb{M}_n(B)) \cong \mathrm{Hom}_{\mathbf{CommAlg}_k}(A_{ab}, B)$. Theorem says that

$$\mathrm{Hom}_{\mathbf{CommAlg}_k}\left(\left(\sqrt[n]{A}\right)_{ab}, B\right) = \mathrm{Hom}_{\mathbf{Alg}_k}\left(\sqrt[n]{A}, i(B)\right) = \mathrm{Hom}_{\mathbf{CommAlg}_k}(A, \mathbb{M}_n(B))$$

□

Proof. We will use the fact that compositions of adjoint functors are adjoint (on the same side).

Let's compose $\mathbb{M}_n$ as follows, define the category of algebras over $\mathbb{M}_n(k)$

$$\mathbf{Alg}_{\mathbb{M}_n(k)} \coloneqq \mathbb{M}_n(k) \downarrow \mathbf{Alg}_k$$

with

- $Ob\left(\mathbf{Alg}_{\mathbb{M}_n(k)}\right) = \left\{\text{homomorphism of algebras } \mathbb{M}_n(k) \xrightarrow{f} A\right\}$
- $Mor\left(\mathbf{Alg}_{\mathbb{M}_n(k)}\right) = \left\{\text{commutatitve triangles } \begin{tikzcd} \mathbb{M}_n(k) \arrow[r, "f"] \arrow[dr, "g"'] & A \arrow[d, "\varphi"] \\ & B \end{tikzcd}\right\}$

Note that for $B \in \mathbf{CommAlg}_k$, $\mathbb{M}_n(B)$ comes with canonical map $\mathbb{M}_n(k) \to \mathbb{M}_n(B)$ by applying $\mathbb{M}_n$ to $k \to B$. Hence we have the following commuting diagram

$$\begin{tikzcd} \mathbf{Alg}_k \arrow[r, "\mathbb{M}_n(k)\otimes_k -"] \arrow[dr, "\mathbb{M}_n(-)"'] & \mathbf{Alg}_{\mathbb{M}_n(k)} \arrow[d, bend left, "U"] \\ & \mathbf{Alg}_k \arrow[u, bend left, "\mathbb{M}_n(k)\sqcup_k"] \end{tikzcd}$$

$\mathbb{M}_n(k) \sqcup_k B$ is the free product (coproduct) of $\mathbb{M}_n(k)$ with B. $\mathbb{M}_n(k) \sqcup_k B = \operatorname{colim}\left[\mathbb{M}_n(k) \leftarrow k \to B\right]$ is the pushout (see later).

The forgetful functor $U : \mathbf{Alg}_{\mathbb{M}_n(k)} \to \mathbf{Alg}_k$ has left adjoint $\mathbb{M}_n(k) \sqcup_k -$. We only need to show that $\mathbb{M}_n(k) \otimes_k - : \mathbf{Alg}_k \to \mathbf{Alg}_{\mathbb{M}_n(k)}$ has left adjoint, which follows from next lemma. $\square$

Exercise 1.6.

1. Any nonzero algebra homomorphism $\mathbb{M}_n(k) \to A$ is injective. This follows from the fact that $\mathbb{M}_n(k)$ is a simple ring.
2. Right ideals in $\mathbb{M}_n(k)$ are is one-to-one correspondence to the Grassmannians $\coprod_{p=0}^{n} Gr(p, n)$.

Proof. The two statements are results of Morita equivalence between k and $\mathbb{M}_n(k)$.

The ideals in $\mathbb{M}_n(k)$ are in 1-1 correspondence to the ideals in k. Since k is a field, $\mathbb{M}_n(k)$ has no nontrivial ideal.

The right ideals are in 1-1 correspondence to the right ideals in k^n as a right k-module (vector space), i.e the subspaces of k^n, which are exactly the elements in the Grassmannians $\coprod_{p=0}^{n} Gr(p,n)$.
□

Lemma 1.7. *The matrix functor $\mathbb{M}_n(k) \otimes_k - : \mathbf{Alg}_k \to \mathbf{Alg}_{\mathbb{M}_n(k)}$ has quasi-inverse given by*

$$\begin{array}{cccc} (-)^{\mathbb{M}_n(k)}: & \mathbf{Alg}_{\mathbb{M}_n(k)} & \longrightarrow & \mathbf{Alg}_k \\ & (f:\mathbb{M}_n(k) \to A) & \longmapsto & A^{\mathbb{M}_n(k)} = \{a \in A \mid [a, f(m)] = 0, \forall m \in \mathbb{M}_n(k)\} \end{array}$$

Proof. We need to show two things

1. $(\mathbb{M}_n(A))^{\mathbb{M}_n(k)} \cong A$

   Consider the elementary matrices $e_{ij} \in \mathbb{M}_n(A)$, $\mathbb{M}_n(A) = \mathrm{Span}\{e_{ij}\}$, so for any $M \in \mathbb{M}_n(A)$, write $M = \sum m_{ij} e_{ij}$. The map $\mathbb{M}_n(k) \to \mathbb{M}_n(A)$ is given by $e^k_{ij} \mapsto e_{ij}$ where e^k_{ij} is the elementary matrix in $\mathbb{M}_n(k)$. Hence $(\mathbb{M}_n(A))^{\mathbb{M}_n(k)} = \left\{M \in (\mathbb{M}_n(A))^{\mathbb{M}_n(k)} \mid [M, e_{ij}] = 0\right\}$. By direct computation, we can get that if $M \in (\mathbb{M}_n(A))^{\mathbb{M}_n(k)}$ then $M = aI$ for some $a \in A$. So we have $(\mathbb{M}_n(A))^{\mathbb{M}_n(k)} \cong A$.

2. $\mathbb{M}_n\left(\mathbb{M}_n(k) \xrightarrow{f} A\right)^{\mathbb{M}_n(k)} \cong \left(\mathbb{M}_n(k) \xrightarrow{f} A\right)$

   The previous exercise shows that f is injective, so we can embed $\mathbb{M}_n(k)$ into A, and $\left(\mathbb{M}_n(k) \xrightarrow{f} A\right)^{\mathbb{M}_n(k)} = \{a \in A \mid [a, f(e)] = 0, \forall e \in \mathbb{M}_n(k)\} = kf(e_{11}) \oplus \cdots \oplus kf(e_{nn})$. Hence

   $$\mathbb{M}_n\left(\left(\mathbb{M}_n(k) \xrightarrow{f} A\right)^{\mathbb{M}_n(k)}\right) = \mathbb{M}_n(k) f(e_{11}) \oplus \cdots \oplus \mathbb{M}_n(k) f(e_{nn}) = \mathbb{M}_n(k) f(1) = f(\mathbb{M}_n(k))$$

   which is exactly $\mathbb{M}_n(k) \xrightarrow{f} A$.

□

This lemma implies that the centralizer functor $(-)^{\mathbb{M}_n(k)}$ is both left and right adjoint of $\mathbb{M}_n(-)$. Hence

$$\sqrt[n]{A} = (\mathbb{M}_n(k) \sqcup_k A)^{\mathbb{M}_n(k)} = \{w = m_1 * a_1 * \cdots * m_i * a_i \in \mathbb{M}_n(k) \sqcup_k A \mid m_i \in \mathbb{M}_n(k), a_i \in A, [w, m] = 0, \forall m \in \mathbb{M}_n$$

Question: What do the elements in $\sqrt[n]{A}$ look like?

Trick. Consider the elementary matrix $e_{ij}, 1 \leq i,j \leq n$, $\mathbb{M}_n(k) = \mathrm{Span}\{e_{ij}\}_{i,j=1}^n$. Recall the relation satisfied by e_{ij}'s,

$$\begin{cases} \sum_{i=1}^n e_{ii} = 1 \\ e_{ij}e_{kl} = \delta_{jk}e_{il} \end{cases} \tag{13}$$

Take $w \in \mathbb{M}_n(k) \sqcup_k A$ any word and define its (i,j)-element by $w_{ij} = \sum_{k=1}^n e_{ki} * w * e_{jk}$. Then

$$[w_{ij}, e_{lm}] = \left(\sum_{k=1}^n e_{ki} * w * e_{jk}\right) e_{lm} - e_{lm}\left(\sum_{k=1}^n e_{ki} * w * e_{jk}\right) = e_{li} * w * e_{jl} - e_{li} * w * e_{jl} = 0$$

$\forall 1 \leq l, m \leq n$ from 13.

$\sqrt[n]{A}$ is generated by a_{ij} where $a \in A \subset \mathbb{M}_n(k) \sqcup_k A$.

$$\sqrt[n]{A} = \frac{k\langle a_{ij} : a \in A, i,j = 1, \cdots, n\rangle}{\langle \text{matrix relation}\rangle}$$

The centralization is given by

$$\begin{array}{ccc} \sqrt[n]{A} & \leftarrow & \mathbb{M}_n(k) \sqcup_k A \\ \|w_{ij}\| & \leftarrow\!\shortmid & w \end{array}$$

Remark 1.12. The matrix algebra $\mathbb{M}_n(k)$ can be replaced with "Azumaya k-algebras" (local matrix algebras) in the key lemma. See notes in [KT].

1.8 Colimits

Fix $\mathcal{J}$ a small category of which we think as "index" category.

Example 1.11.

1. Discrete category $\mathcal{J} = \{\bullet\ \bullet\ \bullet\ \bullet\ \bullet \cdots\}$ with $Mor(\mathcal{J}) = \{\mathrm{Id}_j\}_{i \in Ob(\mathcal{J})}$.

2. Pushout category $\mathcal{J} = \{a \leftarrow b \rightarrow c\}$ with 3 objects and two non-identity morphisms.

3. Sequential category $\mathcal{J} = \{0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots\}$ with $Od(\mathcal{J}) = \mathbb{Z}_+$ and $\mathrm{Hom}_{\mathcal{J}}(i,j) = \begin{cases} \emptyset & i > j \\ \rightarrow & i \leq j \end{cases}$.

4. A poset viewed as a category $\mathcal{J}$.

For a given category $\mathcal{C}$, denote $\mathcal{C}^{\mathcal{J}} = \mathbf{Fun}(\mathcal{J}, \mathcal{C})$. We call a functor $F : \mathcal{J} \to \mathcal{C}$ a diagram of shape $\mathcal{J}$ in $\mathcal{C}$.

There is a constant functor $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{J}}$ with

$$\begin{array}{cccc}
\Delta : & \mathcal{C} & \longrightarrow & \mathcal{C}^{\mathcal{J}} \\
 & X & \longmapsto & \left(\begin{array}{cccc} \Delta(X) : & \mathcal{J} & \to & \mathcal{C} \\ & i & \mapsto & X \\ & (i \to j) & \mapsto & (\mathrm{Id}_X : X \to X) \end{array} \right) \\
 & \left(X \xrightarrow{f} Y\right) & \longmapsto & \left(\begin{array}{c} \Delta(f) : \Delta(X) \to \Delta(Y) \\ \Delta(f) = \{\Delta(f)_i = f : X \to Y\}_{i \in Ob(\mathcal{J})} \end{array} \right)
\end{array}$$

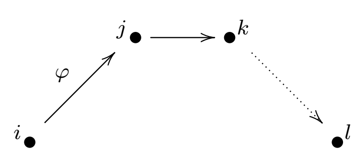
Given any $F \in Ob\left(\mathcal{C}^{\mathcal{J}}\right)$, define the functor

$$\begin{array}{cccc}
\tilde{F} : & \mathcal{C} & \longrightarrow & \mathbf{Set} \\
 & X & \longmapsto & \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(F, \Delta(X))
\end{array}$$

Definition 1.22. If $\tilde{F}$ is (co)representable with (co)representing object in $\mathcal{C}$, this (co)representing object is called *colimit* of F (a.k.a. direct limit, inductive limit) and is denoted by $\mathrm{colim}_{\mathcal{J}}(F)$.

By definition, $\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{J}}(F), X) \cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(F, \Delta(X))$.

Convenient pictures: If $X \in Ob(\mathcal{C})$, the natural transformation $t : F \Rightarrow \Delta(X)$ are given by $t = \{t_i : F(j) \to X\}_{j \in Ob(\mathcal{J})}$. Visualize $\mathcal{J}$ as directed graph



t can be visualized as cone

$$\begin{tikzcd}
 & F(j) \arrow[rr] \arrow[dr, "t_j"] & & F(k) \arrow[dl, "t_k"'] \arrow[dr, dotted] & \\
F(i) \arrow[ur, "F(\varphi)"] \arrow[r, "t_i"'] & X & & & F(l) \arrow[lll, "t_l"]
\end{tikzcd}$$

cone below F with vertex X

$C = \operatorname{colim}_{\mathcal{J}}(F) \in Ob(\mathcal{C})$ comes together with $s : F \Rightarrow \Delta(\operatorname{colim}_{\mathcal{J}} F)$. The universal property of colimits says that $\forall t : F \Rightarrow \Delta(X)$, there exists a unique $f : \operatorname{colim}(F) \to X$ such that $\Delta(f)\, s = t$.

$$\begin{tikzcd}
 & F(j) \arrow[rr] \arrow[dr, dashed, "s_j"] \arrow[ddr, "t_j"'] & & F(k) \arrow[dl, dashed, "s_k"'] \arrow[ddl, "t_k"] \arrow[dr, dotted] & \\
F(i) \arrow[ur, "F(\varphi)"] \arrow[rr, dashed, "s_i"] \arrow[drr, "t_i"'] & & \operatorname{colim}_{\mathcal{J}}(F) \arrow[d, dashed, "\Delta(f)"] & & F(l) \arrow[ll, dashed, "s_l"'] \arrow[dll, "t_l"] \\
 & & X & &
\end{tikzcd}$$

As a consequence of Theorem 1.9, we have the following proposition.

Proposition 1.3. *Suppose colim*$_{\mathcal{J}}(F)$ *exists for every* $F \in Ob\left(\mathcal{C}^{\mathcal{J}}\right)$*, then the assignment* $F \mapsto$ *colim*$_{\mathcal{J}}(F)$ *extends to a functor* $\mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ *which is left adjoint to* Δ.

$$colim_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \rightleftarrows \mathcal{C} : \Delta$$

Definition 1.23. $\mathcal{C}$ is called *cocomplete* if colim exists for all small $\mathcal{J}$ and all $F \in Ob\left(\mathcal{C}^{\mathcal{J}}\right)$.

Example 1.12. $\mathcal{C} = \mathbf{Set}, \mathbf{Top}, \mathbf{Mod}(A)$ are cocomplete.

Let's consider $F : \mathcal{J} \to \mathbf{Set}$. Define $U \coloneqq \coprod_{j \in Ob(\mathcal{J})} F(j) = \{(j, x) : j \in Ob(\mathcal{J}), x \in F(j)\}$, then $\operatorname{colim}_{\mathcal{J}}(F) = U/\sim$ where "$\sim$" is given by $(j, x) \sim (j', x')$ if there exists $\varphi : j \to j'$ such that $F(\varphi)(x) = x'$.

Basic Examples

Coproduct. Discrete category $\mathcal{J} = \{\bullet\ \bullet\ \bullet \cdots \bullet\ \bullet\}$ with $Mor(\mathcal{J}) = \{\mathrm{Id}_j\}_{j \in Ob(\mathcal{J})}$.

A functor $F : \mathcal{J} \to \mathcal{C}$ is equivalent to $F = \{X_j = F(j)\}_{j \in Ob(\mathcal{J})}$.

$\mathrm{colim}_{\mathcal{J}}(F) = \coprod_{j \in Ob(\mathcal{J})} X_j$ is called the *coproduct* (*sum*) of objects X_j in $\mathcal{C}$.

A natural transformation $s : F \Rightarrow \Delta(\mathrm{colim}_{\mathcal{J}}(F))$ is equivalent to $\left\{ s_j : X_j \to \coprod_{j \in Ob(\mathcal{J})} X_j \right\}_{j \in Ob(\mathcal{J})}$.

By convention, if $\mathcal{J} = \emptyset$, there exists a unique $F : \mathcal{J} \to \mathcal{C}$, $\mathrm{colim}_{\mathcal{J}}(F) = \emptyset$ is the initial object in $\mathcal{C}$, i.e. $\forall X \in Ob(\mathcal{C})$ there exists a unique $f : \emptyset \to X$.

Pushout. $\mathcal{J} = \{a \leftarrow b \rightarrow c\}$

A functor $F : \mathcal{J} \to \mathcal{C}$ is given by the diagram $\{X_a \leftarrow X_b \rightarrow X_c\}$ called the pushout data in $\mathcal{C}$.

$\mathrm{colim}_{\mathcal{J}}(F) = \mathrm{colim}\left\{X_a \overset{\alpha}{\leftarrow} X_b \overset{\gamma}{\rightarrow} X_c\right\} =: X_a \coprod_{X_b} X_c =: P \in Ob(\mathcal{C})$ is called *pushout* of the diagram.

P is characterized by the property that

$$\begin{tikzcd}
X_b \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & X_c \arrow[d, "s_\alpha"'] \arrow[ddr, bend left, "t_\alpha"] & \\
X_a \arrow[r, "s_\gamma"] \arrow[drr, bend right, "t_\gamma"'] & P \arrow[dr, dashed, "f"] & \\
 & & Y
\end{tikzcd}$$

is a Cocartesian (commutative) diagram in $\mathcal{C}$, $s_\alpha \gamma = s_\gamma \alpha$, and $\forall t_\alpha : X_c \to Y, t_\gamma : X_a \to Y$ there exists a unique $f : P \to Y$ such that $t_\alpha = f s_\alpha, t_\gamma = f s_\gamma$.

Coequalizer. $\mathcal{J} = \left\{ 0 \bullet \rightrightarrows \bullet\ 1 \right\}$

A functor $F : \mathcal{J} \to \mathcal{C}$ is given by the diagram $\left\{ X_0 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X_1 \right\}$ where $X_i = F(i), i = 0, 1$.

$\mathrm{colim}_{\mathcal{J}}(F) = \mathrm{coeq}\left\{ X_0 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X_1 \right\} =: C \in Ob(\mathcal{C})$ is called a *coequalizer* of the diagram.

C is characterized by the properties

- there exists $\pi : X_1 \to C$ such that $\pi\alpha = \pi\beta$, $X_0 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} X_1 \xrightarrow{\pi} C$.

- $\forall\ X_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X_1 \xrightarrow{\pi} Y$, there exists a unique $f : C \to Y$ such that

$$\begin{tikzcd} X_0 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & X_1 \arrow[rr, "p"] \arrow[dr, "\pi"'] & & Y \\ & & C \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

  commutes. We think of (C, π) as a "non-abelian" generalization of cokernel.

Warning. In general,

$$\operatorname{coeq}\left\{ X_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X_1 \right\} \neq \operatorname{colim}\left\{ X_1 \xleftarrow{\alpha} X_0 \xrightarrow{\beta} X_1 \right\}$$

even when $X_0 = X_1$.

Exercise 1.7. Prove

$$\operatorname{coeq}\left\{ X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X \right\} = \operatorname{colim}\left\{ X \xleftarrow{(\alpha, \mathrm{Id}_X)} X \coprod X \xrightarrow{(\beta, \mathrm{Id}_X)} X \right\}$$

whenever both sides make sense.

Proof. The second copy of X and the second map Id_X in the colimit diagram ensures that $t_\alpha = t_\beta$. $\square$

Sequential Colimits (telescopes). $\mathcal{J} = \mathbb{Z}_+ = \{0 \to 1 \to 2 \to 3 \to \cdots\}$.

A functor $F : \mathcal{J} \to \mathcal{C}$ is equivalent to $\left\{ X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \to \cdots \right\}$, called sequential directed systems in $\mathcal{C}$.

In the categories $\mathbf{Set}, \mathbf{Top}, \mathbf{Mod}\,(A)$, if all $i_j : X_j \hookrightarrow X_{j+1}$ are injective, the diagram is called filtrations on $\operatorname{colim}_{\mathcal{J}} F$.

$\operatorname{colim}_{\mathcal{J}} F \cong \bigcup_{n \geq 0} X_n$.

In the category **Top**, $U \subseteq \operatorname{colim}_{\mathcal{J}}(F)$ is open if and only if $U \cap X_n$ is open in $X_n, \forall n$.

1.9 Limits

Note that $F : \mathcal{J} \to \mathcal{C}$ can be written as $F^{op} : \mathcal{J}^{op} \to \mathcal{C}^{op}$.

Definition 1.24. The *limit* of F is defined as $\lim_{\mathcal{J}}(F) \coloneqq \operatorname{colim}_{\mathcal{J}}(F^{op})$.

Let's list basic properties of limits.

1. If $\lim_{\mathcal{J}}(F)$ exists, it represents the functor

$$\begin{aligned} \tilde{F}: \ \mathcal{C}^{op} &\to \mathbf{Set} \\ X &\mapsto \operatorname{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta(X), F) \end{aligned}$$

or equivalently,

$$\operatorname{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta(X), F) \cong \operatorname{Hom}_{\mathcal{C}}(X, \lim_{\mathcal{J}}(F)).$$

There exists a unique natural transformation $s : \Delta(\lim_{\mathcal{J}} F) \Rightarrow F$ with $s = \{s_j : \lim_{\mathcal{J}} F \to F(j)\}_{j \in Ob(\mathcal{J})}$ such that for any $t : \Delta(X) \Rightarrow F$, pictured as

$$\begin{tikzcd}
 & & X \arrow[dll, "s_i"'] \arrow[ddl, "s_j"] \arrow[ddr, "s_k"'] \arrow[drr, "s_l"] & & \\
F(i) \arrow[dr, "F(\varphi)"] & & & & F(l) \\
 & F(j) \arrow[rr] & & F(k) \arrow[ur, dashed] &
\end{tikzcd}$$

there exists a unique $f : X \to \lim_{\mathcal{J}} F$ such that $s\Delta(f) = t$.

$$\begin{tikzcd}
 & & X \arrow[dll, "t_i"'] \arrow[ddl, "t_j"] \arrow[d, dashed, "\Delta(f)"] \arrow[ddr, "t_k"'] \arrow[drr, "t_l"] & & \\
F(i) \arrow[dr, "F(\varphi)"] & & \lim_{\mathcal{J}} F \arrow[ll, dashed, "s_i"'] \arrow[rr, dashed, "s_l"] \arrow[dl, dashed, "s_j"] \arrow[dr, dashed, "s_k"'] & & F(l) \\
 & F(j) \arrow[rr] & & F(k) \arrow[ur, dashed] &
\end{tikzcd}$$

2. If $\lim_{\mathcal{J}}(F)$ exists for all $F \in Ob\left(\mathcal{C}^{\mathcal{J}}\right)$ then $\Delta : \mathcal{C} \rightleftarrows \mathcal{C}^{\mathcal{J}} : \lim_{\mathcal{J}}$ is a pair of adjoint functors.

Examples

Products. Discrete category $\mathcal{J} = \{\bullet \ \bullet \ \bullet \cdots \bullet \ \bullet\}$ with $Mor(\mathcal{J}) = \{\mathrm{Id}_j\}_{j \in Ob(\mathcal{J})}$.

A functor $F : \mathcal{J} \to \mathcal{C}$ is equivalent to $F = \{X_j = F(j)\}_{j \in Ob(\mathcal{J})}$.

$\lim_{\mathcal{J}}(F) = \prod\limits_{j \in Ob(\mathcal{J})} X_j$ is called the *product* of objects X_j in $\mathcal{C}$ indexed by $j \in \mathcal{J}$.

A natural transformation $s : \Delta(\lim_{\mathcal{J}}(F)) \Rightarrow F$ is equivalent to $\left\{ s_j : \prod\limits_{j \in Ob(\mathcal{J})} X_j \to X_j \right\}_{j \in Ob(\mathcal{J})}$.

By convention, if $\mathcal{J} = \emptyset$, there exists a unique $F : \mathcal{J} \to \mathcal{C}$, $\lim_{\mathcal{J}}(F) = *$ is the terminal object in $\mathcal{C}$, i.e. $\forall X \in Ob(\mathcal{C})$ there exists a unique $f : X \to *$.

Pullbacks. $\mathcal{J} = \{a \to b \leftarrow c\}$

A functor $F : \mathcal{J} \to \mathcal{C}$ is given by the diagram $\{X_a \to X_b \leftarrow X_c\}$.

$\lim_{\mathcal{J}}(F) = \lim\left\{X_a \xrightarrow{\alpha} X_b \xleftarrow{\gamma} X_c\right\} =: X_a \prod\limits_{X_b} X_c \in Ob(\mathcal{C})$ is called *pullback* of the diagram.

$X_a \prod\limits_{X_b} X_c$ is characterized by the property that

$$
\begin{tikzcd}
Y \arrow[dr, dashed, "f"] \arrow[drr, bend left, "t_\gamma"] \arrow[ddr, bend right, "t_\alpha"'] & & \\
& X_a \prod\limits_{X_b} X_c \arrow[r, "p_\gamma"] \arrow[d, "p_\alpha"'] & X_c \arrow[d, "\alpha"] \\
& X_a \arrow[r, "\gamma"] & X_b
\end{tikzcd}
$$

is a Cartesian (commutative) diagram in $\mathcal{C}$, $\gamma p_\alpha = \alpha p_\gamma$, and $\forall t_\gamma : Y \to X_c, t_\alpha : Y \to X_a$ there exists a unique $f : Y \to X_a \prod\limits_{X_b} X_c$ such that $t_\alpha = p_\alpha f, t_\gamma = p_\gamma f$.

p_α is the base change of α along γ, p_γ is the base change of γ along α.

Equalizer. $\mathcal{J} = \left\{ 0 \bullet \rightrightarrows \bullet\, 1 \right\}$

A functor $F : \mathcal{J} \to \mathcal{C}$ is given by the diagram $\left\{ X_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X_1 \right\}$ where $X_i = F(i), i = 0, 1$.

$\lim_{\mathcal{J}}(F) = \mathrm{eq}\left\{ X_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X_1 \right\} =: E \in Ob(\mathcal{C})$ is called a *equalizer* of the diagram.

E is characterized by the properties

- there exists $i : E \to X_0$ such that $\alpha i = \beta i$, $E \xrightarrow{i} X_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X_1$.

- $\forall\, Y \xrightarrow{j} X_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X_1$, there exists a unique $f : Y \to E$ such that

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, dashed, "f"'] & & X_0 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & X_1 \\ & E \arrow[ur, "i"'] & & \end{tikzcd}$$

  commutes.

Exercise 1.8. (Dual version)

$$\mathrm{eq}\left\{ X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X \right\} = \lim \left\{ X \xrightarrow{(\alpha, \mathrm{Id}_X)} X \coprod X \xleftarrow{(\beta, \mathrm{Id}_X)} X \right\}$$

Exercise 1.9. (Origin of Quillen Algebraic K-Theory)

Consider the algebraic K-theory of finite fields $\mathbb{F}_q, q = p^n$. $BU = BGL(\mathbb{C})^{top}$.

$F \to \bigwedge^p F$ induces Adams operation ψ^q.

$$BGL(\mathbb{F}_q) \xrightarrow{f} \mathrm{hofib}_\phi \left(BU \underset{\mathrm{Id}}{\overset{U^q}{\rightrightarrows}} BU \right) \cong \lim \left(BU \xrightarrow{(\psi^q, \mathrm{Id}_X)} BU \coprod BU \xleftarrow{\Delta} BU \right)$$

So $H_*(BGL(\mathbb{F}_q)) \cong H_*(\mathrm{hofib}(\phi))$, and

$$BGL(\mathbb{F}_q) \to BGL(\mathbb{F}_q)^+ .$$

1.10 Properties of (co)limits

Fix a small category $\mathcal{J}$. Let $\mathcal{C}$ be a category that admits all colimits of diagrams of shape $\mathcal{J}$.

Let $F : \mathcal{J} \to \mathcal{C}, G : \mathcal{C} \to \mathcal{D}$ be two functors.

The colimit of F comes together with $s : F \Rightarrow \Delta(\mathrm{colim}_{\mathcal{J}} F), s = \{s_j : F(j) \to \mathrm{colim}_{\mathcal{J}} F\}_{j \in Ob(\mathcal{J})}$, which gives

$$Gs : GF \Rightarrow G\Delta(\mathrm{colim}_{\mathcal{J}} F), Gs = \{G(s_j) : GF(j) \to G(\mathrm{colim}_{\mathcal{J}} F)\}_{i \in Ob(\mathcal{J})}$$

Assume in addition that $\mathrm{colim}_{\mathcal{J}}(GF)$ exists, by UMP of colimits, we have a unique $\alpha_F : \mathrm{colim}_{\mathcal{J}}(GF) \to$

$G\left(\mathrm{colim}_{\mathcal{J}}F\right)$ such that

$$\begin{tikzcd} GF \arrow[rr] \arrow[dr] & & \Delta\left(G\left(\mathrm{colim}_{\mathcal{J}}F\right)\right) \\ & \Delta\left(\mathrm{colim}_{\mathcal{J}}\left(GF\right)\right) \arrow[ur] & \end{tikzcd}$$

commutes.

Definition 1.25. We say that G preserves (or commutes with) colimits of shape $\mathcal{J}$ if for every $F \in Ob\left(\mathcal{C}^{\mathcal{J}}\right)$

1. $\mathrm{colim}_{\mathcal{J}}\left(GF\right)$ exists, and
2. α_F is isomorphism.

Equivalently, using Yoneda Lemma, this can be restated as

Definition 1.26. G preserves (or commutes with) colimits of shape $\mathcal{J}$ if the functor

$$\begin{aligned} \widetilde{GF}: \quad \mathcal{D} &\to \mathbf{Set} \\ X &\mapsto \mathrm{Hom}_{\mathcal{D}^{\mathcal{J}}}\left(GF, \Delta X\right) \end{aligned}$$

is (co)represented by $G\left(\mathrm{colim}_{\mathcal{J}}F\right) \in Ob\left(\mathcal{D}\right)$.

Dually, assume $\mathcal{C}$ admits all limits of shape $\mathcal{J}$.

Definition 1.27. A functor G preserves (or commutes with) limits of shape $\mathcal{J}$ if the functor

$$\begin{aligned} \widetilde{\widetilde{GF}}: \quad \mathcal{D}^{op} &\to \mathbf{Set} \\ X &\mapsto \mathrm{Hom}_{\mathcal{D}^{\mathcal{J}}}\left(\Delta X, GF\right) \end{aligned}$$

is represented by $G\left(\lim_{\mathcal{J}}F\right) \in Ob\left(\mathcal{D}\right)$. Or equivalently, this can be restated as, for every $F \in Ob\left(\mathcal{C}^{\mathcal{J}}\right)$

1. $\lim_{\mathcal{J}}\left(GF\right)$ exists, and
2. $\beta_F : G\left(\lim_{\mathcal{J}} F\right) \to \lim_{\mathcal{J}}\left(GF\right)$ is an isomorphism.

Definition 1.28. A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ maps colimits to limits if the functor

$$\begin{array}{rccc} \widetilde{GF}: & \mathcal{D}^{op} & \to & \mathbf{Set} \\ & X & \mapsto & \mathrm{Hom}_{\mathcal{D}^{\mathcal{J}^{op}}}(\Delta X, GF) \end{array}$$

is represented by $G(\mathrm{colim}_{\mathcal{J}} F) \in Ob(\mathcal{D})$, or equivalently, this can be restated as, for every $F \in Ob(\mathcal{C}^{\mathcal{J}})$

1. $\lim_{\mathcal{J}}(GF)$ exists, and
2. $G(\mathrm{colim}_{\mathcal{J}} F) \to \lim_{\mathcal{J}}(GF)$ is an isomorphism.

Lemma 1.8. *Assume $\mathcal{C}$ admits all limits and colimits of shape $\mathcal{J}$. Then $\forall X \in Ob(\mathcal{C})$,*

1. *$h^X = Hom_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Set}$ preserves limits.*
2. *$h_X = Hom_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \mathbf{Set}$ maps colimits to limits.*

Equivalently,

1. *$Hom_{\mathcal{C}}(X, lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}}(Hom_{\mathcal{C}}(X, -) \circ F) = \lim_{\mathcal{J}} Hom_{\mathcal{C}}(X, F(-))$.*
2. *$Hom_{\mathcal{C}}(colim_{\mathcal{J}} F, X) \cong \lim_{\mathcal{J}}(Hom_{\mathcal{C}}(F(-), X))$.*

Proof. Fix $X \in Ob(\mathcal{C})$.

1. Consider $G = h^X : \mathcal{C} \to \mathbf{Set}$, $GF = h^X \circ F : \mathcal{J} \to \mathbf{Set}$. Since $\mathbf{Set}$ is complete, $S := \lim_{\mathcal{J}}(GF) = \lim_{\mathcal{J}}(\mathrm{Hom}_{\mathcal{C}}(X, F(-))) \in \mathbf{Set}$ exists. By definition, it comes together with

$$s : \Delta S \Rightarrow h^X \circ F, s = \{s_j : S \to \mathrm{Hom}_{\mathcal{C}}(X, F(j))\}_{j \in Ob(\mathcal{J})}$$

Given s_j's is equivalent to given $s : S \to \prod_{j \in Ob(\mathcal{J})} \mathrm{Hom}_{\mathcal{C}}(X, F(j))$. This map is injective and we can identify

$$S \cong \left\{ (f_j)_{j \in Ob(\mathcal{J})} \in \prod_{j \in Ob(\mathcal{J})} \mathrm{Hom}_{\mathcal{C}}(X, F(j)) \,|\, \forall \varphi : i \to j, \quad \begin{tikzcd} & X \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ F(i) \arrow[rr, "F(\varphi)"] & & F(j) \end{tikzcd} \quad \text{commutes} \right\}$$

By UMP of limits, there exists a unique $\bar{f} : X \to \lim_{\mathcal{J}} F$ such that

$$\begin{tikzcd}
& X \arrow[d, "\bar{f}"] \arrow[ddl, bend right, "f_i"'] \arrow[ddr, bend left, "f_j"] & \\
& \lim_{\mathcal{J}} F \arrow[dl, "s_i"'] \arrow[dr, "s_j"] & \\
F(i) \arrow[rr, "F(\varphi)"] & & F(j)
\end{tikzcd}$$

commutes . This defines a map

$$\begin{array}{cccc} \psi : & S & \longrightarrow & \mathrm{Hom}(X, \lim_{\mathcal{J}} F) = h^X(\lim_{\mathcal{J}} F) \\ & (f_j)_{j \in Ob(\mathcal{J})} & \longmapsto & \bar{f} \end{array}$$

ψ is the inverse of $\beta : h^X(\lim_{\mathcal{J}} F) \to S$.

□

Exercise 1.10. Check that ψ is the inverse of β.

Example 1.13. Let $\mathcal{C} = \mathbf{Vect}_k$, k is a field. $\mathcal{J} = \mathbb{Z}_+ = \{0 \to 1 \to 2 \to \cdots\}$ the sequential diagram. A functor $F : \mathcal{J} \to \mathbf{Vect}_k$ is given by $\left\{V_0 \xrightarrow{f_9} V_1 \xrightarrow{f_1} V_2 \to \cdots\right\}$.

Let $\varinjlim V_i := \mathrm{colim}_{\mathcal{J}} F$, and $\varprojlim V_i := \lim_{\mathcal{J}} F$. Then for any $X \in \mathbf{Vect}_k$,

$$\mathrm{Hom}_k\left(X, \varprojlim V_i\right) = \varprojlim \mathrm{Hom}_k(X, V_i),$$

$$\mathrm{Hom}_k\left(\varinjlim V_i, X\right) = \varprojlim \mathrm{Hom}_k(V_i, X).$$

Question. In general when does h^X commutes with $\mathrm{colim}_{\mathcal{J}}$?

Exercise 1.11. Prove that in $\mathcal{C} = \mathbf{Vect}_k$, $\mathrm{Hom}_k\left(X, \varinjlim V_i\right) \cong \varinjlim \mathrm{Hom}_k(X, V_i)$ if and only if $\dim_k X < \infty$.

Remark 1.13. See notes [DG] on DG categories, section on small (compact) objects.

Definition 1.29. Fix a small category $\mathcal{J}$. In a category $\mathcal{C}$ we can define $X \in Ob(\mathcal{C})$ to be *$\mathcal{J}$-small* if $\alpha : \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{colim}_{\mathcal{J}} F) \xrightarrow{\cong} \mathrm{colim}_{\mathcal{J}}(\mathrm{Hom}_{\mathcal{C}}(X, F(-)))$ is an isomorphism for any $F \in Ob\left(\mathcal{C}^{\mathcal{J}}\right)$.

Question. Let A be a ring, $\mathcal{C} = \mathbf{Mod}(A)$ be the category of left or right modules over A. What are the small modules in $\mathcal{C}$ for $\mathcal{J} = \mathbb{Z}_+$.

Theorem 1.13. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors, then F preserves colimits and G preserves limits (whenever they exists).*

Proof. Consider $H : \mathcal{J} \to \mathcal{C}$ such that $\mathrm{colim}_{\mathcal{J}}(H)$ exists in $\mathcal{C}$. Take any $Y \in Ob(\mathcal{D})$ and

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(F(\mathrm{colim}_{\mathcal{J}}(H)), Y) &\cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{J}}(H), GY) \\
&\cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(H(-), GY) \\
&\cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{D}}(FH(-), Y) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}_{\mathcal{J}}(FH), Y)
\end{aligned}$$

Similarly, consider $I : \mathcal{J} \to \mathcal{C}$ such that $\lim_{\mathcal{J}}(H)$ exists in $\mathcal{D}$. Take any $Z \in Ob(\mathcal{C})$

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(Z, G(\lim_{\mathcal{J}}(H))) &\cong \mathrm{Hom}_{\mathcal{D}}(FZ, \lim_{\mathcal{J}}(H)) \\
&\cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{D}}(FZ, H(-)) \\
&\cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(Z, GH(-)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(Z, \lim_{\mathcal{J}}(GH))
\end{aligned}$$

□

1.11 All "important" concepts are Kan Extensions

Problem. Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, we want to extend F along G. Namely, we want to find a functor $H : \mathcal{E} \to \mathcal{D}$ so that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \\
\mathcal{E} \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

"commutes" up to isomorphism, that is $F \cong H \circ G$. In general, such H does not exist for several reasons.

Obstructions: For $X, Y \in \mathrm{Ob}(\mathcal{C})$, consider the Hom sets

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, hook, "F"] \arrow[d, "G"'] & \mathrm{Hom}_{D}(FX,FY) \\ \mathrm{Hom}_{\mathcal{E}}(GX,GY) & \end{tikzcd}$$

with F faithful and G not faithful. Then there might exist $f_1, f_2 : X \to Y$ such that $Ff_1 \neq Ff_2$ in $\mathcal{D}$ but $Gf_1 = Gf_2$ in $\mathcal{E}$. In this case such H will not exists. It may also happen that there exist $X, Y \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{D}}(FX,FY) = \emptyset$ and $\mathrm{Hom}_{\mathcal{E}}(GX,GY) \neq \emptyset$.

Example. Homotopic Functors.

Let $F : \mathbf{Top}_* \to \mathbf{Ab}$ be a functor from pointed topological spaces to abelian groups and let $G : \mathbf{Top}_* \to \mathbf{Ho}(\mathbf{Top}_*)$ be the localization functor. In general, an extension of F along the localization functor does not exist. If such H exists, F is called *homotopic.*

Example. Induction and Coinduction.

Consider groups $H \leq G$ and think of H and G as categories with one object $\{*\}$. Write $\mathbf{G}$ for the category associated to the group G. The inclusion map $H \hookrightarrow G$ induces a functor $i : \mathbf{H} \hookrightarrow \mathbf{G}$. A representation (ρ, V) of H in vector spaces defines a functor

$$\begin{array}{cccc} \rho : & \mathbf{H} & \longrightarrow & \mathbf{Vect}_k \\ & * & \longmapsto & V \\ & \left(* \xrightarrow{g} *\right) & \mapsto & \rho_g \in \mathrm{End}_k(V). \end{array}$$

Furthermore, we have the induction and coinduction functors $\mathbf{G} \to \mathbf{Vect}_k$ which are defined from classical representation theory as

$$\mathrm{Ind}_\rho(*) := k[G] \otimes_{k[H]} V$$
$$\mathrm{Coind}_\rho(*) := \mathrm{Hom}_{k[H]}(k[G], V).$$

In both cases, it is clear that $\mathrm{Ind}_\rho \circ i \ncong \rho$ and $\mathrm{Coind}_\rho \circ i \ncong \rho$.

Instead of insisting on having such an isomorphism, we look for universal morphisms in two ways, either from F or to F. That is, we approximate F by universal morphisms $\eta : F \Rightarrow HG$ or

$\varepsilon : HG \Rightarrow F$.

Definition 1.30. A *left Kan extension* of F along G is a pair consisting of a functor $L_GF : \mathcal{E} \to \mathcal{D}$ together with a morphism $\eta : F \Rightarrow L_GF \circ G$ satisfying the universal mapping property: For all $H : \mathcal{E} \to \mathcal{D}$ and $\gamma : F \to HG$, there exists a unique $\varphi : L_GF \Rightarrow H$ such that $\gamma = \varphi G \circ \eta$, where φG is the horizontal composition

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] & \mathcal{E} \arrow[r, bend left, "L_GF"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathcal{D} \end{tikzcd}$$

given by $\varphi G = \{\varphi_{GX} : L_GF(GX) \to H(GX)\}_{X \in \mathrm{Ob}(\mathcal{C})}$. The UMP means that the pair (L_GF, η) is initial among all pairs $(H : \mathcal{E} \to \mathcal{D}, \gamma : F \Rightarrow HG)$.

Dually, we have a right Kan extension.

Definition 1.31. A *right Kan extension* of F along G is a pair consisting of a functor $R_GF : \mathcal{E} \to \mathcal{D}$ together with a morphism $\varepsilon : R_GF \circ G \Rightarrow F$ which is couniversal among all pairs $(H : \mathcal{E} \to \mathcal{D}, \delta : HG \Rightarrow F)$. That is, for any such pair (H, δ), there exists a unique morphism $\varphi : H \Rightarrow R_GF$ such that $\varepsilon \circ \varphi G = \delta$ where φG is the horizontal composition

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] & \mathcal{E} \arrow[r, bend left, "R_GF"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Uparrow\varphi"] & \mathcal{D} \end{tikzcd}$$

given by $\varphi G = \{\varphi_{GX} : H(GX) \to R_GF(GX)\}_{X \in \mathrm{Ob}(\mathcal{C})}$. One can encode this data in the following non-commuting diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Uparrow\varepsilon", near start] & \mathcal{D} \\ \mathcal{E} \arrow[ur, "R_GF"'] \arrow[ur, bend right=60, "H"'] \arrow[ur, phantom, "\Uparrow\exists!\delta", near end] & \end{tikzcd} .$$

Another way to define a Kan extension is as follows: Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, define

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, - \circ G) : \mathbf{Fun}(\mathcal{E}, \mathcal{D}) \longrightarrow \mathbf{Set}$$

by $H \mapsto \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, H \circ G)$ on objects. The Left Kan extension $L_GF \in \mathrm{Ob}(\mathbf{Fun}(\mathcal{E}, \mathcal{D}))$ is precisely the object representing this functor since the UMP precisely gives the isomorphism

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, H \circ G) \cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}(L_GF, H)$$

for all $H \in \mathrm{Ob}\left(\mathbf{Fun}\left(\mathcal{E},\mathcal{D}\right)\right)$.

Proposition 1.4. *For a fixed $G : \mathcal{C} \to \mathcal{E}$, if $L_G F$ exists for all $F : \mathcal{C} \to \mathcal{D}$, then there is an adjoint pair*

$$L_G\left(-\right) : \mathbf{Fun}\left(\mathcal{C},\mathcal{D}\right) \rightleftharpoons \mathbf{Fun}\left(\mathcal{E},\mathcal{D}\right) : G_* := \left(-\right) \circ G.$$

Dually, if $R_G F$ exists for all F, the right Kan extension can be realized as a right adjoint to G_*.

Remark. Since L_G is realized in the proposition as a left adjoint to G_*, it is an approximation of F from the left and is hence a *left* Kan extension. Furthermore, if both $L_G F$ and $R_G F$ exist for all $F : \mathcal{C} \to \mathcal{D}$ and a fixed $G : \mathcal{C} \to \mathcal{E}$, then we obtain the adjoint triple

$$\begin{tikzcd}
\mathbf{Fun}\left(\mathcal{E},\mathcal{D}\right) \arrow[d, "G_*"'] \\
\mathbf{Fun}\left(\mathcal{C},\mathcal{D}\right) \arrow[u, bend left=60, "L_G(-)"] \arrow[u, bend right=60, "R_G(-)."']
\end{tikzcd}$$

Example. Group Representations.

Let G be a discrete group and k a field. Let $\mathrm{Rep}_k\left(G\right)$ be the category of k-linear representations of G and G-equivariant linear maps. Let $\mathbf{G}$ denote the category with one object $\{*\}$ and $\mathrm{Hom}_{\mathbf{G}}\left(*,*\right) = G$. Then $\mathrm{Rep}_k\left(G\right)$ can be identified with the category $\mathbf{Fun}\left(\mathbf{G},\mathbf{Vect}_k\right)$. A representation $\rho : G \to GL\left(V\right)$ of G induces the functor $\widetilde{\rho}$ mapping the object $*$ to V and any morphism $g : * \to *$ to $\rho_g : V \to V$. Suppose $H \leq G$ is a subgroup, there are classical functors relating $\mathrm{Rep}_k\left(H\right)$ and $\mathrm{Rep}_k\left(G\right)$: Res^G_H which is precisely post-composition with the inclusion functor $\mathbf{i} : \mathbf{H} \hookrightarrow \mathbf{G}$, $\mathrm{Ind}^G_H := k[G] \otimes_{k[H]} -$, and $\mathrm{Coind}^G_H := \mathrm{Hom}_{k[H]}\left(k[G],-\right)$. Notice that for all $V \in \mathrm{Rep}_k\left(H\right)$ and all $W \in \mathrm{Rep}_k\left(G\right)$, the Hom-Tensor adjunction gives

$$\mathrm{Hom}_G\left(\mathrm{Ind}^G_H\left(V\right),W\right) \cong \mathrm{Hom}_G\left(k\left[G\right] \otimes_{k[H]} V, W\right) \cong \mathrm{Hom}_H\left(V, \mathrm{Hom}_{k[G]}\left(k\left[G\right],W\right)\right) \cong \mathrm{Hom}_H\left(V,W\right)$$

where the last isomorphism arises from the isomorphism $\mathrm{Hom}_{k[G]}\left(k\left[G\right],W\right) \cong \mathrm{Res}^G_H\left(W\right) \cong W$ of H-representations. In this way, Ind^G_H is realized as a left adjoint to Res^G_H. Dually, the isomorphism

$\mathrm{Res}_H^G(W) \cong k[G] \otimes_{k[H]} W$ together with the Hom-Tensor adjunction gives

$$\mathrm{Hom}_H\left(W \otimes_{k[H]} k[G], V\right) \cong \mathrm{Hom}_G\left(W, \mathrm{Hom}_{k[H]}(k[G], V)\right) \cong \mathrm{Hom}_G\left(W, \mathrm{Coind}_H^G(V)\right)$$

which realizes Coind_H^G as a right adjoint to Res_H^G. Identifying $\mathrm{Rep}_k(G)$ and $\mathrm{Rep}_k(H)$ with the functor categories $\mathbf{Fun}(\mathbf{G}, \mathbf{Vect}_k)$ and $\mathbf{Fun}(\mathbf{H}, \mathbf{Vect}_k)$ respectively, $\mathrm{Res}_H^G = - \circ \mathbf{i} =: \mathbf{i}_*$ and we obtain the adjunction

$$\begin{tikzcd} \mathbf{Fun}(\mathbf{G}, \mathbf{Vect}_k) \arrow[d, "\mathbf{i}_*"] \\ \mathbf{Fun}(\mathbf{H}, \mathbf{Vect}_k) \arrow[u, bend left=60, "L_{\mathbf{i}}(-)"] \arrow[u, bend right=60, "R_{\mathbf{i}}(-)"'] \end{tikzcd}$$

where $L_{\mathbf{i}} = \mathrm{Ind}_H^G$ and $R_{\mathbf{i}} = \mathrm{Coind}_H^G$.

Example. Colimits and Limits.

Colimits can be interpreted as left Kan extensions and limits as right Kan extensions. Let us consider $F : \mathcal{C} \to \mathcal{D}$ be any functor. Take $*$ to be the terminal category consisting of one object and only the identity morphism. Consider

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \\ * \arrow[ur, "H"'] & \end{tikzcd}$$

and notice that for a functor $H : * \to \mathcal{D}$ and natural transformation $\gamma : F \Rightarrow HG$, since H picks out a single object object in $\mathcal{D}$, $HG \cong \mathbf{\Delta}(X)$ where $X = H(*) \in \mathrm{Ob}(\mathcal{D})$ and $\mathbf{\Delta} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$ is the constant functor. In particular, any morphism $\gamma : F \Rightarrow HG$ is precisely $\gamma : F \Rightarrow \mathbf{\Delta}(X)$ for some $X \in \mathrm{Ob}(\mathcal{D})$. In this case, the UMP for the Left Kan extension of F along G is precisely the UMP for $\mathrm{Colim}(F)$. Dually, $R_G F$ is realized as $\mathrm{Lim}(F)$.

Example. Adjunctions.

Let $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$ be a pair of adjoint functors. Then we have the unit and counit morphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$. The unit η realizes G as a left Kan extension of $\mathrm{Id}_{\mathcal{C}}$ along F and the counit ε realizes F as a right Kan extension of $\mathrm{Id}_{\mathcal{D}}$ along G.

Lemma 1.9. *Left adjoint functors preserve left Kan extensions.*

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ has a Left Kan extension $(L_G F, \eta)$ along $G : \mathcal{C} \to \mathcal{E}$. Given an adjoint pair $L : \mathcal{D} \rightleftharpoons \mathcal{B} : R$ with unit $\iota : \mathrm{Id}_{\mathcal{D}} \to RL$ and counit $\nu : LR \to \mathrm{Id}_{\mathcal{C}}$, consider the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[r, shift left, "L"] & \mathcal{B} \arrow[l, shift left, "R"] \\ \mathcal{E} \arrow[ur, "L_G F"] \arrow[urr, bend right, "L_G(LF)"'] & & \end{tikzcd} .$$

By UMP for Left Kan extensions, there exists a natural morphism $\alpha : L_G(LF) \to L \circ L_G F$. The lemma is precisely saying that provided L has a right adjoint, α is an isomorphism.

For any $H : \mathcal{E} \to \mathcal{B}$, we have isomorphisms of sets

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{B})}(L \circ L_G F, H) &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{B})}(L_G F, RH) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, RHG) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{B})}(LF, HG) \end{aligned}$$

where the first and third isomorphism is due to the adjunction $L^* : \mathbf{Fun}(\mathcal{E}, \mathcal{D}) \rightleftharpoons \mathbf{Fun}(\mathcal{E}, \mathcal{B}) : R^*$ of the pre-composition functors F^* and R^* coming from the adjunction of the pair (L, R) and the second isomorphism is from the UMP for left Kan extensions. By Yoneda lemma, $L \circ L_G F \cong L_G(LF)$. Take $H = L \circ L_G F$ and consider the image of $\mathrm{Id}_{L \circ L_G F}$ under the above isomorphisms

$$\mathrm{Id}_{L \circ L_G F} \mapsto \iota_{L_G F} \mapsto \iota_{L_G F \circ G} \circ \eta \mapsto L\eta.$$

More precisely, $(L \circ L_G F, L\eta)$ is also Left Kan extension of LF along G. $\square$

This lemma suggests that $L_G F$ can be expressed as the colimit of some natural diagrams. In fact, there are two natural diagrams for which $L_G F$ (or equivalently $R_G F$) can be expressed as a colimit (or limit): (co)slice and (co)end diagrams.

Slice Categories and Pointwise Kan Extensions

Given a functor $G : \mathcal{C} \to \mathcal{E}$ and a fixed object $e \in \mathrm{Ob}(\mathcal{E})$, define the slice category over e, denoted G/e, by $\mathrm{Ob}(G/e) = \{(c, f) : c \in \mathrm{Ob}(\mathcal{C}), f : Gc \to e\}$ and $\mathrm{Hom}_{G/e}\left((c, f), \left(c', f'\right)\right)$ to be the set consisting of morphisms $\varphi \in \mathrm{Hom}_{\mathcal{C}}\left(c, c^{'}\right)$ making the following diagram commute

$$\begin{tikzcd} Gc \arrow[r, "G\varphi"] \arrow[d, "f"'] & Gc^{'} . \\ d \arrow[ur, "f^{'}"'] & \end{tikzcd}$$

The slice category G/e comes together with a forgetful functor $U : G/e \to \mathcal{C}$, mapping $(c, f) \mapsto c$ and forgetting the commutative property of the morphisms. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, consider the diagram $G/e \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}$. Assuming the colimit of FU exists, define $\widetilde{L_G F}e :=$ colim $\left\{G/e \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right\}$. Any morphism $\varphi : e \to e'$ in $\mathcal{E}$ induces a functor $\varphi_* : G/e \to G/e^{'}$ mapping $(c, f) \mapsto (c, \varphi \circ f))$. Moreover, we have a commutative diagram

$$\begin{tikzcd} G/e \arrow[r, "U"] \arrow[d, "\varphi_*"'] & \mathcal{C} \arrow[r, "F"] \arrow[d, equal] & \mathcal{D} \arrow[d, equal] \\ G/e^{'} \arrow[r, "U^{'}"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

thereby obtaining a morphism of colimits $\varphi_* : \widetilde{L_G F}e \to \widetilde{L_G F}e^{'}$. If $\widetilde{L_G F}e$ exists for all $e \in \mathrm{Ob}\,(\mathcal{E})$, then $\widetilde{L_G F}$ defines a functor $\widetilde{L_G F} : \mathcal{E} \to \mathcal{D}$.

Proposition 1.5. *Under the above assumptions, we have a natural isomorphism of functors $\widetilde{L_G F} \cong L_G F$.*

The proposition follows from a direct comparison of the UMP for colimits and Left Kan extensions. As a consequence, we obtain the explicit formula

$$L_G F e = \mathrm{colim}\left\{G/e \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right\} \tag{14}$$

whenever they exist.

Definition 1.32. If colim $\left\{G/e \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right\}$ exists for all $e \in \mathrm{Ob}\,(\mathcal{E})$ so that $L_G F$ is given by equation 14 above, then $L_G F$ is called a *pointwise* Left Kan extension.

Remark. If $\mathcal{D}$ is cocomplete, then any left Kan extension of a functor whose target is $\mathcal{D}$ is automatically pointwise. In practice however, $\mathcal{D}$ is not always cocomplete (for example, $\mathbf{Ho}\,(\mathcal{C})$ for a model category $\mathcal{C}$ or $\mathcal{D}(\mathcal{A})$ for an abelian category $\mathcal{A}$ are rarely cocomplete). Still, for some "good" pairs (F, G), Left Kan extensions may happen to be pointwise.

Definition 1.33. A left Kan extension is called *absolute* if for all $H : \mathcal{D} \to \mathcal{B}$ in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[r, "H"] & \mathcal{B} \\ \mathcal{E} \arrow[ur, "L_G F"] \arrow[urr, bend right, "L_G(LF)"'] & & \end{tikzcd}$$

the natural morphism $\alpha : L_G(HF) \to H \circ L_G F$ is an isomorphism. Namely, every such H preserves $L_G F$.

Exercise 1.12. Show that any absolute Kan extension is pointwise.

Exercise 1.13. Consider the following diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] \arrow[d, "G"'] & \mathcal{D} \arrow[l, shift left, "R"] \arrow[d, "F"] \\ \mathcal{E} \arrow[r, shift left, "R_G(FL)"] & \mathcal{E}' \arrow[l, shift left, "L_F(GR)"] \end{tikzcd} .$$

If $L_F(GR)$ and $R_G(FL)$ exist and both are absolute, then they are adjoint. An example of this is when $\mathcal{C}$ and $\mathcal{D}$ are model categories and G and F are localization functors, then any Quillen pair (L, R) satisfies this condition, namely, $(\mathbb{L}L, \mathbb{R}R)$ is an adjoint pair

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] \arrow[d, "\mathrm{loc}"'] & \mathcal{D} \arrow[l, shift left, "R"] \arrow[d, "\mathrm{loc}"] \\ \mathbf{Ho}(\mathcal{C}) \arrow[r, shift left, "\mathbb{L}L"] & \mathbf{Ho}(\mathcal{D}) \arrow[l, shift left, "\mathbb{R}R"] \end{tikzcd} .$$

Corollary 1.7. *Assume $L_G F$ is pointwise and G is fully faithful. Then the natural map $\eta : F \xrightarrow{\sim} L_G F \circ G$ is an isomorphism of functors.*

Proof. Take any $c \in \mathrm{Ob}(\mathcal{C})$ and consider the comma category G/Gc. Since G is fully faithful, G/Gc has a terminal object $* = (c, \mathrm{Id}_{Gc})$. Note also that if $\mathcal{J}$ has a terminal object $*$, then for any diagram $F : \mathcal{J} \to \mathcal{D}$, $\mathrm{colim}_{\mathcal{J}}(F) = F(*)$. In particular, for any $c \in \mathrm{Ob}(\mathcal{C})$, $L_G F(Gc) = \mathrm{colim}\left(G/Gc \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right) = F(U(c, \mathrm{Id}_{Gc})) = Fc$. Hence, $L_G F \circ G \cong F$. $\square$

Coends and Kan extensions

Coends are special kinds of colimits defined as follows: Assume $\mathcal{D}$ is a cocomplete category with arbitrary coproducts and $\mathcal{C}$ is a small category. Given a bifunctor $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, define the coend of S by

$$\int^{c \in \mathrm{Ob}(\mathcal{C})} S(c,c) := \mathrm{Coeq}\left\{ \coprod_{\substack{f: c \to d \\ f \in \mathrm{Mor}(\mathcal{C})}} S(d,c) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c \in \mathrm{Ob}(\mathcal{C})} S(c,c) \right\}$$

where $f^* = S(f, \mathrm{Id}) : S(d,c) \to S(c,c)$ and $f_* = S(\mathrm{Id}, f) : S(d,c) \to S(d,d)$. By UMP for colimits, a coend $X := \int^{c \in \mathrm{Ob}(\mathcal{C})} S(c,c)$ comes together with a family of morphisms $\{\varphi_c : S(c,c) \to X\}_{c \in \mathrm{Ob}(\mathcal{C})}$ making the diagram

$$\begin{tikzcd} S(d,c) \arrow[r, "f^*"] \arrow[d, "f_*"'] & S(c,c) \arrow[d, "\varphi_c"] \\ S(d,d) \arrow[r, "\varphi_d"] & X \end{tikzcd}$$

$F : I \to \mathcal{C}$

coprod + coeq.

commute and is initial among all such pairs.

We can extend some natural constructions as coends of some bifunctors.

Example. Colimits as coends. $I \to \mathcal{C}$

Take a functor $F : \mathcal{C} \to \mathcal{D}$ and define a bifunctor $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ by $\left(c', c\right) \mapsto Fc$ and $\left(f', f\right) \mapsto Ff$ which is constant on the first argument. Then

$I^{op} \times I \to \mathcal{C}$

$(i', i) \mapsto F(i)$

$$\int^{c \in \mathrm{Ob}(\mathcal{C})} S(c,c) = \mathrm{Coeq}\left\{ \coprod_{\substack{f: c \to d \\ f \in \mathrm{Mor}(\mathcal{C})}} Fc \underset{f_* = Ff}{\overset{f^* = \mathrm{Id}}{\rightrightarrows}} \coprod_{c \in \mathrm{Ob}(\mathcal{C})} Fc \right\} \cong \mathrm{Colim}_{\mathcal{C}} F.$$

Indeed, $X := \int^{c \in \mathrm{Ob}(\mathcal{C})} S(c,c)$ comes together with a family of maps $\{\varphi_c : Fc \to X\}_{c \in \mathrm{Ob}(\mathcal{C})}$ such that for all morphisms $f : c \to c'$ in $\mathcal{D}$ the following diagram

$$\coprod_{\substack{f: i \to j \\ f \in \mathrm{Mor}\, I}} F_i \underset{F(f)}{\overset{id}{\rightrightarrows}} \coprod_{i \in \mathrm{Obj}} F_i$$

$y \cong x \oplus z$

$$\begin{tikzcd} Fc \arrow[rr, "Ff"] \arrow[dr, "\varphi_c"'] & & Fc' \arrow[dl, "\varphi_{c'}"] \\ & X & \end{tikzcd}$$

$X = \mathrm{coeq}.$

commutes and is initial among all such families. This is precisely the UMP for the colimit of F.

Example. Functor tensor products.

$\{\varphi_i : F_i \to X\}$ s.t.

$\forall\ i \xrightarrow{f} j$

$$\begin{tikzcd} F_i \arrow[rr, "F(f)"] \arrow[dr, "\varphi_i"'] & {} \arrow[d, phantom, "\circlearrowleft"] & F_j \arrow[dl, "\varphi_j"] \\ & X & \end{tikzcd}$$

commutes. initial.

Let R be an associative ring with unity and let $M \in \mathrm{Ob}(\mathbf{Mod}{-}R)$ and $N \in \mathrm{Ob}(R{-}\mathbf{Mod})$. Consider R as a category $\mathbf{R}$ with a single object $\{*\}$ and $\mathrm{Hom}_{\mathbf{R}}(*,*) = R$. A right R module M can be thought of as a functor $M : \mathbf{R}^{op} \to \mathbf{Ab}$ mapping $* \mapsto M$ and $R \ni r \mapsto (R_r : M \to M, m \mapsto m.r)$. Similarly, a left R module N is a functor $N : \mathbf{R} \to \mathbf{Ab}$ mapping $* \mapsto N$ and $R \ni r \mapsto (L_r : N \to N, n \mapsto r.n)$. Define a bifunctor $S := M \otimes_{\mathbb{Z}} N : \mathbf{R}^{op} \times \mathbf{R} \to \mathbf{Ab}$ by $(*,*) \mapsto M \otimes_{\mathbb{Z}} N$ and $\left(r, r^{'}\right) \mapsto R_r \otimes L_{r^{'}}$. Then

$$\int^{*} M \otimes_{\mathbb{Z}} N = \mathrm{Coeq}\left\{ \bigoplus_{r \in R} M \otimes_{\mathbb{Z}} N \underset{1 \otimes L_r}{\overset{R_r \otimes 1}{\rightrightarrows}} M \otimes_{\mathbb{Z}} N \right\} \cong \frac{M \otimes_{\mathbb{Z}} N}{\langle mr \otimes n - m \otimes rn : \forall r \in R, m \in M, n \in N \rangle}$$

hence $\int^{*} M \otimes_{\mathbb{Z}} N \cong M \otimes_R N$.

We can generalize this construction to tensor products of functors. Let $\mathcal{C}$ be a small category and R an associative ring. Let $M : \mathcal{C}^{op} \to \mathbf{Mod}{-}R$ and $N : \mathcal{C} \to R{-}\mathbf{Mod}$ be functors. Such an M is usually called a *right $(\mathcal{C}, R)$ module* and N a *left $(\mathcal{C}, R)$ module.* Define

$$S := M \boxtimes_R N : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Ab}$$

by $\left(c^{'}, c\right) \mapsto Mc^{'} \otimes_R Nc$ on objects and by $\left(c^{'} \xrightarrow{f^{'}} d^{'}, c \xrightarrow{f} d\right) \mapsto \left(S(f^{'}, f) : Md^{'} \otimes_R Nc \to Mc^{'} \otimes_R Nd\right)$ on morphisms where $S\left(f^{'}, f\right) = Mf^{'}(-) \otimes_R Nf(-), m^{'} \otimes n \mapsto Mf^{'}(m^{'}) \otimes_R Nf(n)$. Define the *functor tensor product* by

$$M \otimes_{\mathcal{C},R} N := \int^{c \in \mathrm{Ob}(\mathcal{C})} M \boxtimes_R N \cong \bigoplus_{c \in \mathrm{Ob}(\mathcal{C})} Mc \otimes_R Nc / U$$

where U is the subgroup of $\bigoplus_{c \in \mathrm{Ob}(\mathcal{C})} Mc \otimes_R Nc$ generated by $Mf(m^{'}) \otimes_R n - m^{'} \otimes_R Nf(n)$ for all $\left(f : c \to c^{'}\right) \in \mathrm{Mor}(\mathcal{C}), m^{'} \in Mc^{'}$, and $n \in Nc$.

Note that taking $R = \mathbb{Z}$ and $\mathcal{C} = \mathbf{R}$ we recover the original tensor product of modules, namely, $M \otimes_{\mathbf{R},\mathbb{Z}} N = M \otimes_R N$. Furthermore, taking M to be the constant functor $M = R : \mathcal{C}^{op} \to \mathbf{Mod}{-}R$ mapping every object to R and every morphism to the identity morphism. Then $R \boxtimes_R N : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Ab}$ is defined by $(c^{'}, c) \mapsto R \otimes_R Nc \cong Nc$ on objects and $(f^{'}, f) \mapsto Nf$ on morphism. In particular, since the bifunctor $R \boxtimes_R N$ is constant on the first argument, $R \otimes_{\mathcal{C},R} N \cong \mathrm{Colim}(N)$.

Exercise 1.14. Fix $A \in \mathrm{Ob}(\mathcal{C})$ and consider $h^A := R[\mathrm{Hom}_{\mathcal{C}}(A, -)] : \mathcal{C} \to R{-}\mathbf{Mod}$ to be the composition of $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathbf{Set}$ and the free left module functor $R[-] : \mathbf{Set} \to R\!-\!\mathbf{Mod}$ and $h_A := R^{op}[\mathrm{Hom}_{\mathcal{C}}(-,A)] : \mathcal{C}^{op} \to \mathbf{Mod}\!-\!R$ defined similarly. Given $M : \mathcal{C}^{op} \to \mathbf{Mod}\!-\!R$ and $N : \mathcal{C} \to R\!-\!\mathbf{Mod}$, show that $h_A \otimes_{\mathcal{C},R} N \cong N(A)$ and $M \otimes_{\mathcal{C},R} h^A \cong M(A)$.

Remark. The functor tensor product defines a bifunctor

$$- \otimes_{\mathcal{C},R} - : \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Mod}\!-\!R) \times \mathbf{Fun}(\mathcal{C}, R\!-\!\mathbf{Mod}) \longrightarrow \mathbf{Ab}$$

which is right exact and balanced and therefore has a left derived functor $(M,N) \mapsto \mathrm{Tor}_*^{\mathcal{C},R}(M,N)$.

Left Kan Extensions as Coends: Given $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, the goal is to compute $L_G F$ in terms of coends. Namely, we need to define a bifunctor $S_e : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ for each $e \in \mathrm{Ob}(\mathcal{E})$ whose coend will compute $L_G F e$.

Fix $e \in \mathrm{Ob}(\mathcal{E})$ and define S_e on objects by

$$\left(c, c'\right) \mapsto \mathrm{Hom}_{\mathcal{E}}\left(Gc', e\right) \cdot Fc =: \coprod_{i : Gc' \to e} (Fc)_i\,,$$

a coproduct of copies of Fc indexed by elements of $\mathrm{Hom}_{\mathcal{E}}\left(Gc', e\right)$ and on morphisms

$$\left(c' \xrightarrow{f'} d', c \xrightarrow{f} d\right) \mapsto \left[S_e\left(f', f\right) : \coprod_{i : Gc' \to e} (Fc)_i \longrightarrow \coprod_{j : Gd' \to e} (Fd)_j\right]$$

where $S_e\left(f', f\right)|_i : (Fc)_i \to (Fd)_{i \circ Gf'}$. Then S_e defines a bifunctor.

Theorem 1.14. *If $\mathcal{C}$ is a cocomplete category with arbitrary coproducts, then $L_G F e \cong \int^{c \in Ob(\mathcal{C})} Hom_{\mathcal{E}}\left(Gc', e\right) \cdot Fc$.*

Example. Induced representations.

Let $H \leq G$ be groups and $\rho : H \to \mathrm{Aut}_k V$ a representation of H. Consider G and H as one-object categories $\mathbf{G}$ and $\mathbf{H}$, then $\rho : \mathbf{H} \to \mathbf{Vect}_k$ is a functor defined by $* \mapsto V$ on objects and $h \mapsto \rho(h)$ on morphisms. Then the induced representation of ρ can realized as a left Kan extension of ρ along the inclusion functor $i : \mathbf{H} \hookrightarrow \mathbf{G}$. Using coends, we can explicitly compute $L_i(\rho)$ as follows

$$L_i(\rho) = \mathrm{Coeq}\left\{ \bigoplus_{h\in H} k[G]\otimes_k V \underset{f_*}{\overset{f^*}{\rightrightarrows}} k[G]\otimes_k V \right\}$$

where $f^*|_h : k[G]\otimes_k V \to k[G]\otimes_k V$ is defined by $g\otimes v \mapsto g\otimes\rho(h)\,v$ and $f_*|_h : k[G]\otimes_k V \to k[G]\otimes_k V$ is defined by $g\otimes v \mapsto gh\otimes v$. In particular, $L_i(\rho) \cong k[G]\otimes_k V/\langle g\otimes\rho(h)\,v - gh\otimes v\rangle \cong k[G]\otimes_{k[H]} V$.

Example. Geometric Realization.

Let $\mathbf{\Delta}$ be the cosimplicial category and let $X_* : \mathbf{\Delta}^{op} \to \mathbf{Top}$ be a "good" simplicial space. Here "good" means that geometric realization will work well, or *Reedy cofibrant* spaces, see section 2.3 in [KT]. The geometric realization $|X_*|$ is defined as follows: Consider the geometric simplex, namely, the functor $\Delta^* : \mathbf{\Delta} \to \mathbf{Top}$, defined by $[n] \mapsto \Delta^n = \left\{(x_0,\ldots,x_n)\in\mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1, x_i \geq 0\right\}$ on objects and mapping a morphism $[n] \xrightarrow{f} [m]$ to $f^* : \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}, e_i \mapsto e_{f(i)}$. Let $h : \mathbf{\Delta} \hookrightarrow \mathbf{Fun}(\mathbf{\Delta^{op}},\mathbf{Set}) =: \mathbf{sSet}$ be the Yoneda functor and $\mathbf{i} : \mathbf{sSet} \hookrightarrow \mathbf{sTop}$ the natural inclusion. Consider $G = \mathbf{i}\circ h$ and define

$$|X_*| := L_G(\Delta^*)(X_*)\,.$$

Let $\Delta_*[n]$ be the image of $[n]$ under the Yoneda functor $h : \mathbf{\Delta} \hookrightarrow \mathbf{sSet}$, then $|\Delta_*[n]| \cong \Delta^n$.

Example. Derived Functors. Let $\mathcal{C}$ be a category with a "nice" class of morphisms $W \subset \mathrm{Mor}(\mathcal{C})$. Morally speaking, we want W to contain all isomorphisms in $\mathcal{C}$ and to be closed under compositions. Then, we can formally define the category $\mathcal{C}[W]^{-1}$, called the *localization of* $\mathcal{C}$ *at* W. The category $\mathcal{C}[W]^{-1}$ is characterized by the UMP: Given any functor $F : \mathcal{C} \to \mathcal{D}$ inverting every $f \in W$, that is Ff is an isomorphism in $\mathcal{D}$, there is a unique functor $\overline{F} : \mathcal{C}[W]^{-1} \to \mathcal{D}$ making commutative the following diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{D} \\ \mathcal{C}[W]^{-1} \arrow[ur, "\overline{F}"'] & \end{tikzcd}$$

where $Q : \mathcal{C} \to \mathcal{C}[W]^{-1}$ is the localization functor, the functor so that $\left(\mathcal{C}[W]^{-1}, Q\right)$ solves this universal problem.

A simple example is when $\mathcal{C}$ is the category $\mathbf{Com}(\mathbf{Ab})$ of complexes of abelian groups. Take W to be the class of *quasi-isomorphisms,* namely, morphisms of complexes $f : K_\bullet \to M_\bullet$ so that the induced map on homology $H_\bullet(f) : H_\bullet(K_\bullet) \to H_\bullet(M_\bullet)$ is an isomorphism. An example of quasi-isomorphisms come from short exact sequences as follows. Consider the short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ of abelian groups and define complexes $K_\bullet = \left[0 \to A \xrightarrow{i} B \to 0\right]$ and $M_\bullet = [0 \to 0 \to C \to 0]$. Consider the map $f : K_\bullet \to M_\bullet$ of complexes given by

$$\begin{tikzcd} K_\bullet \arrow[d, "f"'] & 0 \arrow[r] \arrow[d, "0"] & A \arrow[r, "i"] \arrow[d, "0"] & B \arrow[r] \arrow[d, "p"] & 0 \arrow[d, "0"] \\ M_\bullet & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \end{tikzcd} .$$

Then $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is exact if and only if f is a quasi-isomorphism. The category $\mathbf{Com}(\mathbf{Ab})[W]^{-1} =: \mathcal{D}(\mathbf{Ab})$ is called the (unbounded) derived category.

Idea of derived functors: Consider a functor $F : \mathcal{C} \to \mathcal{D}$ and a class $W \subset \mathrm{Mor}(\mathcal{C})$ of "nice" morphisms in $\mathcal{C}$.

Definition 1.34. A left (respectively right) derived functor of F is the right (respectively left) Kan extension of F along the localization functor $Q : \mathcal{C} \to \mathcal{C}[W]^{-1}$, denoted $\mathbf{R}F := L_Q F$ and $\mathbf{L}F := R_Q F$.

In practice, both categories $\mathcal{C}$ and $\mathcal{D}$ carry some classes $W_\mathcal{C} \subset \mathrm{Mor}(\mathcal{C})$ and $W_\mathcal{D} \subset \mathrm{Mor}(\mathcal{D})$ of "nice" morphisms and we want a functor $\overline{F} : \mathcal{C}[W_\mathcal{C}]^{-1} \to \mathcal{D}[W_\mathcal{D}]^{-1}$ making the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q_\mathcal{C}"'] & \mathcal{D} \arrow[d, "Q_\mathcal{D}"'] \\ \mathcal{C}[W_\mathcal{C}]^{-1} \arrow[r, dotted, "\overline{F}"] & \mathcal{D}[W_\mathcal{D}]^{-1} \end{tikzcd}$$

commutative. If $F(W_\mathcal{C}) \nsubseteq W_\mathcal{D}$, then such an $\overline{F}$ does not exist. The idea is to replace $\overline{F}$ by left or right Kan extensions. Specifically, we define the (total) left (and right) derived functors by the following rule

$$\mathbb{L}F := R_{Q_\mathcal{C}}(Q_\mathcal{D} \circ F) \qquad \mathbb{R}F := L_{Q_\mathcal{C}}(Q_\mathcal{D} \circ F) .$$

This definition is due to Quillen.

Part III

Classical Homological Algebra

Outline:

1. Additive/Abelian categories
2. Classical derived functors: definitions and functors
3. Examples: Tor, Ext, Sheaf (co)homology

2 Additive Categories

2.1 Additive Categories

Definition 2.1. A category $\mathcal{A}$ is *preadditive* ($\mathbb{Z}$-category) if

AB1 $\forall X, Y \in Ob(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ has a structure of an abelian group and the composition map is biadditive, i.e. it factors as follows.

$$\circ : \begin{tikzcd} \mathrm{Hom}_{\mathcal{A}}(X,Y) \times \mathrm{Hom}_{\mathcal{A}}(Y,Z) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_{\mathcal{A}}(X,Z) \\ & \mathrm{Hom}_{\mathcal{A}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{A}}(Y,Z) \arrow[ur] & \end{tikzcd}$$

Definition 2.2. A preadditive category $\mathcal{A}$ is called additive if

AB2 $\mathcal{A}$ is pointed, i.e. there exists an initial object $\emptyset_{\mathcal{A}}$ and a terminal object $*_{\mathcal{A}}$ and they coincide.

AB3 $\mathcal{A}$ has pointwise (hence finite) products, i.e. $\forall X, Y \in Ob(\mathcal{A})$, $X \times Y$ exists.

Notation: We call $\emptyset_{\mathcal{A}} = *_{\mathcal{A}} = 0_{\mathcal{A}}$ the null object in $\mathcal{A}$.

$$\mathrm{Hom}_{\mathcal{A}}(0_{\mathcal{A}}, X) = \mathrm{Hom}_{\mathcal{A}}(X, 0_{\mathcal{A}}) = 0, \forall X \in Ob(\mathcal{A}).$$

Exercise 2.1. Show that the following categories are additive.

1. **Ab** the category of abelian groups.
2. $R-$ **Mod** and **Mod** $-R$ the category of left or right modules over R, where R is any ring.
3. **VB** (X) the category of vector bundles over a topological space X.
4. **Sh** (X) the category of abelian sheaves on a variety X.
5. **Qcoh** (X) the category quasi-coherent sheaves on a variety X.

Exercise 2.2. If $\mathcal{A}$ is additive and $\mathcal{C}$ is small, then $\mathbf{Fun}(\mathcal{C},\mathcal{A})$ is additive.

Solution. The composition of natural transformations

$$\begin{tikzcd}
\mathcal{C}\bullet \arrow[rr, bend left, "F_1"] \arrow[rr, "F_2" description] \arrow[rr, bend right, "F_3"'] \arrow[rr, phantom, shift left=3, "\Downarrow\alpha"] \arrow[rr, phantom, shift right=3, "\Downarrow\beta"] & & \bullet\mathcal{A}
\end{tikzcd}$$

is given by

$$\begin{tikzcd}
F_1A \arrow[r, "\alpha_A"] \arrow[d, "F_1f"'] & F_2A \arrow[r, "\beta_A"] \arrow[d, "F_2f"'] & F_3A \arrow[d, "F_3f"'] \\
F_1B \arrow[r, "\alpha_B"] & F_2B \arrow[r, "\beta_B"] & F_3B
\end{tikzcd}$$

Since $\mathcal{A}$ is additive, the horizontal composition map is biadditive, which shows that the composition of natural transformations is also biadditive.

The initial object in $\mathbf{Fun}(\mathcal{C},\mathcal{A})$ is the constant functor $\Delta(0): C\mapsto 0$, which is also the terminal object.

The product of F_1 and F_2 is $F_1\times F_2$, given by $(F_1\times F_2)C := F_1C\times F_2C$, and similarly $F_1\bigsqcup F_2$ is given by $(F_1\bigsqcup F_2)C := F_1C\bigsqcup F_2C$. Since $\mathcal{A}$ is additive,

$$(F_1\times F_2)C = F_1C\times F_2C = F_1C\bigsqcup F_2C = \left(F_1\bigsqcup F_2\right)C, \forall C\in\mathcal{C}.$$

Therefore $F_1\times F_2 = F_1\bigsqcup F_2$.

Exercise 2.3. Let $\mathcal{A}$ be a category with a single object $*$, $\mathcal{A}$ is a $\mathbb{Z}$-category if and only if $\mathrm{Hom}_{\mathcal{A}}(*,*)$ is an associative ring with 1. In this case $\mathcal{A}$ is not additive.

Proof. Notice that the multiplication in $\mathrm{Hom}_{\mathcal{A}}(*,*)$ is given by morphism composition. $\square$

Lemma 2.1. *In any additive category, a finite product is always a coproduct, i.e.* $X \sqcup Y \cong X \times Y$.

Proof. Recall that $X \times Y$ represents the functor

$$\begin{aligned} \mathcal{A}^{op} &\longrightarrow \mathbf{Set} \\ Z &\longmapsto \mathrm{Hom}_{\mathcal{A}}(Z,X) \times \mathrm{Hom}_{\mathcal{A}}(Z,Y) \end{aligned}$$

i.e. there exists a natural bijection

$$\psi : \mathrm{Hom}_{\mathcal{A}}(Z, X \times Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(Z,X) \times \mathrm{Hom}_{\mathcal{A}}(Z,Y), \forall Z \in Ob(\mathcal{A}). \tag{15}$$

Take $Z = X \times Y$ and consider

$$\psi(\mathrm{Id}_{X\times Y}) = (p_X : X \times Y \to X, p_Y : X \times Y \to Y) \tag{16}$$

By Yoneda lemma, we can express ψ in terms of p_X and p_Y,

$$\psi(f : Z \to X \times Y) = (p_X \circ f, p_Y \circ f)$$

Define $i_X : X \to X \times Y$ by $i_X := \psi^{-1}(\mathrm{Id}_X, 0)$ where $0 : X \to Y$ is the zero morphism and $i_Y : Y \to X \times Y$ by $i_Y := \psi^{-1}(0, \mathrm{Id}_Y)$ where $0 : Y \to X$ is the zero morphism.

Notice by the equation 16 we have the relation $\psi(i_X) = (\mathrm{Id}_X, 0)$ which implies

$$p_X \circ i_X = \mathrm{Id}_X \quad p_Y \circ i_X = 0 \tag{17}$$

and $\psi(i_Y) = (0, \mathrm{Id}_Y)$ which implies

$$p_X \circ i_Y = 0 \quad p_Y \circ i_Y = \mathrm{Id}_Y \tag{18}$$

In addition, we have

$$p_X\left(i_Xp_X+i_Yp_Y\right)=\left(p_Xi_X\right)p_X+\left(p_Xi_Y\right)p_Y=p_X+0=p_X$$
$$p_Y\left(i_Xp_X+i_Yp_Y\right)=\left(p_Yi_X\right)p_X+\left(p_Yi_Y\right)p_Y=0+p_Y=p_Y$$

and it follows that

$$i_Xp_X+i_Yp_Y=\mathrm{Id}_{X\times Y} \tag{19}$$

For any $Z\in Ob\left(\mathcal{A}\right)$ we construct mutually bijections

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{A}}\left(X\times Y,Z\right) & \longleftarrow & \mathrm{Hom}_{\mathcal{A}}\left(X,Z\right)\times\mathrm{Hom}_{\mathcal{A}}\left(Y,Z\right)\\ f\circ p_X+g\circ p_Y & \longleftarrow\!\shortmid & \left(f,g\right)\\ \varphi:X\times Y\to Z & \longmapsto & \left(\varphi\circ i_X,\varphi\circ i_Y\right)\end{array}$$

This follows from the relations 17, 18 and 19.

Hence $X\times Y$ corepresents

$$\begin{array}{ccc}\mathcal{A} & \longrightarrow & \mathbf{Set}\\ Z & \longmapsto & \mathrm{Hom}_{\mathcal{A}}\left(X,Z\right)\times\mathrm{Hom}_{\mathcal{A}}\left(Y,Z\right)\end{array}$$

so $X\bigsqcup Y$ exists and $X\bigsqcup Y\cong X\times Y$. □

Notation. We will use "$\oplus$" for "$\times=\bigsqcup$". Thus $X\oplus Y$ comes together with 4 maps i_X,i_Y,p_X,p_Y satisfying the relations 17, 18 and 19.

Exercise 2.4. Show that the relations 17, 18 and 19 characterize $X\oplus Y$ uniquely up to unique isomorphism, i.e. given $Z\in Ob\left(\mathcal{A}\right)$ with

$$p'_X:Z\to X,p'_Y:Z\to Y,i'_X:X\to Z,i'_Y:Y\to Z$$

satisfying the relations 17, 18 and 19, there exists a unique $\phi:Z\xrightarrow{\sim}X\oplus Y$ such that the following four diagrams commutes.

$$\begin{tikzcd} Z \arrow[rr, "p'_X"] \arrow[dr, "\phi"'] & & X \\ & X\oplus Y \arrow[ur, "p_X"'] & \end{tikzcd}$$

$$\begin{tikzcd} Z \arrow[rr, "p'_Y"] \arrow[dr, "\phi"'] & & Y \\ & X \oplus Y \arrow[ur, "p_Y"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[rr, "i_X"] \arrow[dr, "i'_X"'] & & X \oplus Y \\ & Z \arrow[ur, "\phi"] & \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[rr, "i_Y"] \arrow[dr, "i'_Y"'] & & X \oplus Y \\ & Z \arrow[ur, "\phi"] & \end{tikzcd}$$

Proof. Using the UMP of $X \oplus Y$ there exists a unique $\phi : Z \to X \oplus Y$ such that

$$\begin{tikzcd} Z \arrow[rr, "p'_X"] \arrow[dr, "\phi"'] & & X \\ & X \oplus Y \arrow[ur, "p_X"'] & \end{tikzcd}$$

$$\begin{tikzcd} Z \arrow[rr, "p'_Y"] \arrow[dr, "\phi"'] & & Y \\ & X \oplus Y \arrow[ur, "p_Y"'] & \end{tikzcd}$$

and a unique $\varphi : X \oplus Y \to Z$ such that

$$\begin{tikzcd} X \arrow[rr, "i'_X"] \arrow[dr, "i_X"'] & & Z \\ & X \oplus Y \arrow[ur, "\varphi"] & \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[rr, "i'_Y"] \arrow[dr, "i_Y"'] & & Z \\ & X \oplus Y \arrow[ur, "\varphi"] & \end{tikzcd}$$

Then

$$\mathrm{Id}_Z = i'_X p'_X + i'_Y p'_Y = \varphi i_X p_X \phi + \varphi i_Y p_Y \phi = \varphi \left(i_X p_X + i_Y p_Y \right) \phi = \varphi \phi$$

and

$$\mathrm{Id}_X = p'_X i'_X = p_X \phi \varphi i_X$$

$$\mathrm{Id}_Y = p'_Y i'_Y = p_Y \phi \varphi i_Y$$

$$0 = p'_X i'_Y = p_X \phi \varphi i_Y$$

$$0 = p'_Y i'_X = p_Y \phi \varphi i_X$$

Hence

$$\mathrm{Id}_{X\oplus Y} = i_X p_X + i_X p_Y + i_Y p_X + i_Y p_Y = i_X \left(p_X \phi \varphi i_X\right) p_X + i_X \left(p_X \phi \varphi i_Y\right) p_Y + i_Y \left(p_Y \phi \varphi i_X\right) p_X + i_Y \left(p_Y \phi \varphi i_Y\right) p_Y = i_X p$$

Hence $\phi : Z \to X \oplus Y$ is an isomorphism with inverse φ. □

Exercise 2.5. Give an example showing that infinite coproduct does not coincide with product (even if they both exists).

Proof. In $R-\mathbf{Mod}$, given an infinite index category I, the coproduct is

$$\coprod_{i\in I} X_i = \left\{ (x_i)_{i\in I} \,|x_i \in X_i, \text{only finitely many nonzero } x_i \right\}$$

but the product is

$$\prod_{i\in I} X_i = \left\{ (x_i)_{i\in I} \,|x_i \in X_i \right\}$$

□

Exercise 2.6. Show that

1. $\mathcal{A}$ is additive if and only if $\mathcal{A}^{op}$ is additive.

2. If $\mathcal{A}, \mathcal{B}$ are additive, then $\mathcal{A} \times \mathcal{B}$ is additive.

Remark 2.1. Additivity (of a category) is an intrinsic property, not an extra structure (unlike "triangulated" categories). Indeed, if in any category $\mathcal{C}$ with finite products and coproducts, there are two natural maps associated to each $X \in Ob(\mathcal{C})$

$$\begin{aligned} \Delta_X : X \to X \times X, \Delta_X &\coloneqq \mathrm{Id}_X \times \mathrm{Id}_X && \text{diagonal map} \\ \nabla_X : X \coprod X \to X, \nabla_X &\coloneqq \mathrm{Id}_X \coprod \mathrm{Id}_X && \text{codiagonal or folding map} \end{aligned}$$

Exercise 2.7. Show that the additive structure on each $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is given by

$$\begin{aligned} + : \quad \mathrm{Hom}_{\mathcal{A}}(X,Y) \times \mathrm{Hom}_{\mathcal{A}}(X,Y) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(X,Y) \\ (f_1, f_2) &\longmapsto f_1 + f_2 = \nabla_Y \circ (f_1 \oplus f_2) \circ \Delta_X \end{aligned}$$

Proof. The diagonal map $\Delta_X : X \to X \oplus X$ satisfies $p_k \Delta_X = \mathrm{Id}_X$ for $k = 1, 2$. The folding map $\nabla_X : X \oplus X \to X$ satisfies $\nabla_X i_k = \mathrm{Id}_X$ for $k = 1, 2$.

Given any two maps $f_1, f_2 : X \to Y$, there exists a unique morphism $f_1 \oplus f_2 : X \oplus X \to Y \oplus Y$ such that $p_l \circ (f_1 \oplus f_2) i_k = \delta_{kl} f_k$.

We can therefore define $f_1 + f_2 = \nabla_Y \circ (f_1 \oplus f_2) \circ \Delta_X$. This is associative and commutative. $f + 0 = f$ because $f \oplus 0 = i_1 \circ f \circ p_1$. It's also bilinear. □

$\mathcal{A}$ is additive if and only if

- $\mathcal{A}$ is pointed, and
- $\mathcal{A}$ has finite products and coproducts and they coincide.
- Each $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is additive with respect to "+" defined in the exercise 2.7.

2.2 Additive Functors

Let $\mathcal{A}, \mathcal{B}$ be two additive categories.

Definition 2.3. A functor $F : \mathcal{A} \to \mathcal{B}$ is additive if for any $X, Y \in \mathcal{A}$, $F : \mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ is a homomorphism of abelian groups.

Example 2.1. (Additive functors)

1. forgetful functor $R - \mathbf{Mod} \to \mathbf{Ab}$.
2. (Continuous) section functors $\Gamma : \mathbf{VB}(X) \to \mathbf{Sh}(X)$, where $\mathbf{VB}(X)$ is the category of vector bundles on a space X and $\mathbf{Sh}(X)$ is the category of abelian sheaves on X.

3. $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

4. Let M_R be a right R-module, the functor

$$\begin{array}{rccc} M \otimes_R - : & R - \mathbf{Mod} & \to & \mathbf{Ab} \\ & N & \mapsto & M \otimes_R N \end{array}$$

Example 2.2. Restriction/extension of scalars.

Given a ring homomorphism $f : A \to B$, we denote the triple of additive functors $\left(f^*, f_*, f^!\right)$, (resp. extension, restriction, coinduction)

$$\begin{tikzcd} B - \mathbf{Mod} \arrow[d, "f_*"] \\ A - \mathbf{Mod} \arrow[u, bend left=60, "f^*"] \arrow[u, bend right=60, "f^!"'] \end{tikzcd}$$

We have

$$\begin{array}{rccc} f^* : & A - \mathbf{Mod} & \to & B - \mathbf{Mod} \\ & M & \mapsto & B \otimes_A M \end{array}$$

when we consider B as a B, A-bimodule ${}_B B_A$, whose right A-module structure is given by $b \cdot a = b \cdot f(a)$. And

$$\begin{array}{rccc} f^! : & A - \mathbf{Mod} & \to & B - \mathbf{Mod} \\ & M & \mapsto & \mathrm{Hom}_{\mathcal{A}}\left({}_A B_B, M\right) \end{array}$$

where the B-module structure on $\mathrm{Hom}_{\mathcal{A}}\left({}_A B_B, M\right)$ is given by $b \cdot f(m) = f(m \cdot b)$.

Recall the tensor-hom adjunction ${}_A M, {}_B N, {}_B \Omega_A$

$$\begin{array}{ccc} \mathrm{Hom}_B\left(\Omega \otimes_A M, N\right) & \cong & \mathrm{Hom}_A\left(M, \mathrm{Hom}_B\left(\Omega, N\right)\right) \\ \left(f : \Omega \otimes_A M \to N\right) & \longmapsto & \left(\begin{array}{ccc} M & \to & \mathrm{Hom}_B\left(\Omega, N\right) \\ m & \mapsto & \left(\begin{array}{ccc} \Omega & \to & N \\ \omega & \mapsto & f\left(\omega \otimes m\right) \end{array}\right) \end{array}\right) \end{array}$$

is an isomorphism. As a consequence,

$$\mathrm{Hom}_B\left(f^*(M), N\right) = \mathrm{Hom}_B\left(B \otimes_A M, N\right) \cong \mathrm{Hom}_A\left(M, \mathrm{Hom}_B\left(B, N\right)\right) = \mathrm{Hom}_A\left(M, f_*(N)\right)$$

On the other hand, we can identify $f_*(N) \cong {}_A B_B \otimes_B N$, and we have

$$\mathrm{Hom}_A\left(f_*(N), M\right) \cong \mathrm{Hom}_A\left(B \otimes_B N, M\right) \cong \mathrm{Hom}_B\left(N, \mathrm{Hom}_A\left(B, M\right)\right) = \mathrm{Hom}_B\left(N, f^!(M)\right)$$

Example 2.3. Nonadditive functors between additive categories: Tensor, symmetric and exterior powers.

Let R be a commutative ring, for $n \geq 2$,

$$\begin{array}{rccc} \otimes^n: & R-\mathbf{Mod} & \to & R-\mathbf{Mod} \\ & M & \mapsto & \underbrace{M \otimes_R \cdots \otimes_R M}_{n} \end{array}$$

$$\begin{array}{rccc} \mathrm{Sym}_R^n: & R-\mathbf{Mod} & \to & R-\mathbf{Mod} \\ & M & \mapsto & \mathrm{Sym}_R^n(M) = \dfrac{M^{\otimes n}}{\left\langle m_1 \otimes \cdots \otimes m_n - m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(n)} \mid m_i \in M \right\rangle_{\sigma \in S_n}} \end{array}$$

$$\begin{array}{rccc} \bigwedge_R^n: & R-\mathbf{Mod} & \to & R-\mathbf{Mod} \\ & M & \mapsto & \bigwedge_R^n(M) = \dfrac{M^{\otimes n}}{\left\langle m_1 \otimes \cdots \otimes m_n - \mathrm{sgn}(\sigma) m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(n)} \mid m_i \in M \right\rangle_{\sigma \in S_n}} \end{array}$$

Lemma 2.2. *If $\mathcal{A}, \mathcal{B}$ are two additive categories, $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, then there exists a natural isomorphism*

$$F(X \otimes Y) \cong F(X) \otimes F(Y), \forall X, Y \in Ob(\mathcal{A}).$$

Proof. Recall $X \oplus Y$ is an object in $\mathcal{A}$ characterized uniquely by specifying 4 morphisms

$$\begin{array}{ll} p_X : X \oplus Y \to X & p_Y : X \oplus Y \to Y \\ i_X : X \to X \oplus Y & i_Y : Y \to X \oplus Y \end{array}$$

satisfying the relations 17, 18 and 19. Applying F to the four morphisms p_X, p_Y, i_X, i_Y, we get

$F(X \oplus Y) \in Ob(\mathcal{B})$ together with $F(p_X), F(p_Y), F(i_X), F(i_Y)$ which also satisfies the relations 17, 18 and 19, hence by uniqueness

$$F(X \otimes Y) \cong F(X) \otimes F(Y), \forall X, Y \in Ob(\mathcal{A}).$$

$\square$

If $\mathcal{A}, \mathcal{B}$ are additive categories, then the additive functor $F : \mathcal{A} \to \mathcal{B}$ form a (full) subcategory $\mathbf{Fun}_{Add}(\mathcal{A}, \mathcal{B}) \subseteq \mathbf{Fun}(\mathcal{A}, \mathcal{B})$.

Exercise 2.8. Show that $\mathbf{Fun}_{\mathrm{Add}}(\mathcal{A}, \mathcal{B})$ is strictly full, i.e. if $F \cong F'$, $F \in \mathbf{Fun}_{\mathrm{Add}}(\mathcal{A}, \mathcal{B})$, then $F' \in \mathbf{Fun}_{\mathrm{Add}}(\mathcal{A}, \mathcal{B})$.

Proof. Let $\alpha : F \Rightarrow F'$ be the natural isomorphism between F and F'. For any $X, Y \in Ob(\mathcal{A})$, and $f, g : X \to Y$ in $\mathcal{A}$, we have the following commuting diagrams

$$\begin{tikzcd} FX \arrow[r, "\alpha_X", "\cong"'] \arrow[d, "Ff"'] & F'X \arrow[d, "F'f"] \\ FY \arrow[r, "\alpha_Y", "\cong"'] & F'Y \end{tikzcd}$$

$$\begin{tikzcd} FX \arrow[r, "\alpha_X", "\cong"'] \arrow[d, "Fg"'] & F'X \arrow[d, "F'g"] \\ FY \arrow[r, "\alpha_Y", "\cong"'] & F'Y \end{tikzcd}$$

$$\begin{tikzcd} FX \arrow[r, "\alpha_X", "\cong"'] \arrow[d, "F(f+g)"'] & F'X \arrow[d, "F'(f+g)"] \\ FY \arrow[r, "\alpha_Y", "\cong"'] & F'Y \end{tikzcd}$$

Since F is additive, $F(f+g) = Ff + Fg$, so

$$F'f + F'g = \alpha_Y \circ Ff \circ \alpha_X^{-1} + \alpha_Y \circ Fg \circ \alpha_X^{-1} = \alpha_Y \circ (Ff + Fg) \circ \alpha_X^{-1} = \alpha_Y \circ F(f+g) \circ \alpha_X^{-1} = F'(f+g),$$

i.e. F' is additive. $\square$

$\mathbf{Fun}_{\mathrm{Add}}(\mathcal{A}, \mathcal{B})$ is an additive category. See nice example below.

Exercise 2.9. Show that a natural equivalence of categories (assume both $\mathcal{A}, \mathcal{B}$ are small, though not necessary)

$$\mathbf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \cong \mathbf{Fun}(\mathcal{A}, \mathbf{Fun}(\mathcal{B}, \mathcal{C}))$$

restricts to an equivalence of categories

$$\mathbf{Fun}_{\text{Add}}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \cong \mathbf{Fun}_{\text{Add}}(\mathcal{A}, \mathbf{Fun}_{\text{Add}}(\mathcal{B}, \mathcal{C}))$$

Proof. The equivalence of categories

$$\mathbf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \cong \mathbf{Fun}(\mathcal{A}, \mathbf{Fun}(\mathcal{B}, \mathcal{C}))$$

is given by

$$\begin{array}{rccc} \varphi : & \mathbf{Fun}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) & \cong & \mathbf{Fun}(\mathcal{A}, \mathbf{Fun}(\mathcal{B}, \mathcal{C})) \\ & (F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}) & \mapsto & \left(\begin{array}{rccc} \varphi(F) : & \mathcal{A} & \to & \mathbf{Fun}(\mathcal{B}, \mathcal{C}) \\ & X & \mapsto & F(X, -) \end{array} \right) \end{array}$$

If F is additive, i.e. F is a biadditive functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$, then $\varphi(F)(X) = F(X, -) : \mathcal{B} \to \mathcal{C}$ is additive because for any $g_1, g_2 : Y_1 \to Y_2$,

$$\varphi(F)(X)(g_1 + g_2) = F(X, -)(g_1 + g_2) = F(\text{Id}_X, g_1 + g_2) = F(\text{Id}_X, g_1) + F(\text{Id}_X, g_2) = \varphi(F)(X)(g_1) + \varphi(F)(X)$$

and $\varphi(F) : X \mapsto F(X, -)$ is additive because for any $f_1, f_2 : X_1 \to X_2$, $g : Y_1 \to Y_2$,

$$\varphi(F)(f_1 + f_2)(g) = F(f_1 + f_2, g) = F(f_1, g) + F(f_2, g) = \varphi(F)(f_1)(g) + \varphi(F)(f_2)(g).$$

□

Note that if $\mathcal{A}$ is additive, then for each $X \in Ob(\mathcal{A})$,

$$h_X = \text{Hom}_{\mathcal{A}}(-, X) : \mathcal{A}^{op} \to \mathbf{Set}$$

$$h^X = \text{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathbf{Set}$$

factor through

$$h_X = \mathrm{Hom}_{\mathcal{A}}(-, X) : \mathcal{A}^{op} \to \mathbf{Ab}$$

$$h^X = \mathrm{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \to \mathbf{Ab}$$

Exercise 2.10. Show that $h_X \in \mathbf{Fun}_{\mathrm{Add}}(\mathcal{A}^{op}, \mathbf{Ab})$ and $h^X \in \mathbf{Fun}_{\mathrm{Add}}(\mathcal{A}, \mathbf{Ab})$ and the Yoneda functors

$$\begin{array}{rccc} h_* : & \mathcal{A} & \to & \mathbf{Fun}_{\mathrm{Add}}(\mathcal{A}^{op}, \mathbf{Ab}) \\ & X & \mapsto & h_X \end{array}$$

$$\begin{array}{rccc} h^* : & \mathcal{A} & \to & \mathbf{Fun}_{\mathrm{Add}}(\mathcal{A}, \mathbf{Ab}) \\ & X & \mapsto & h^X \end{array}$$

are additive.

Proof. It suffices to prove for h_X and dualize to get h^X. For any $f, g : Y \to Z$ in $\mathcal{A}$

$$\begin{array}{rccl} h_X(f+g) = (f+g)^* : & \mathrm{Hom}_{\mathcal{A}}(Z, X) & \to & \mathrm{Hom}_{\mathcal{A}}(Y, X) \\ & h & \mapsto & (f+g)^*(h) = h \circ (f+g) = h \circ f + h \circ g = f^*(h) + g^*(h) = (f^* + g \end{array}$$

we have $h_X(f+g) = (f+g)^* = f^* + g^* = h_X(f) + h_X(g)$. Hence h_X is additive.

For $f, g : X \to Y$,

$$\begin{array}{rccc} h_{f+g} : & h_X = \mathrm{Hom}_{\mathcal{A}}(-, X) & \to & h_Y = \mathrm{Hom}_{\mathcal{A}}(-, Y) \\ & h & \mapsto & (f+g) \circ h = f \circ h + g \circ h = h_f(h) + h_g(h) = (h_f + h_g)(h) \end{array}$$

we have $h_{f+g} = h_f + h_g$, so h_* is additive. $\square$

Lemma 2.3. *Let $\mathcal{A}, \mathcal{B}$ be additive, $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ is an adjoint pair, then F is additive if and only if G is additive.*

Proof. Assume that F is additive, consider the natural bijection associated to (F, G), then

$$\psi : \mathrm{Hom}_{\mathcal{A}}(X, GY) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{B}}(FX, Y)$$

is an isomorphism of abelian groups. Indeed, ψ can be expressed as

$$\psi\left(f : X \to GY\right) = \left(\eta_Y \circ F\left(f\right) : FX \xrightarrow{Ff} FGY \xrightarrow{\eta_Y} Y\right)$$

where $\eta : FG \to \mathrm{Id}_{\mathcal{D}}$ is the counit of (F, G).

Note that $f \mapsto Ff$ is additive because F is additive, and $\phi \mapsto \eta_Y \circ \phi$ is additive because $\mathcal{B}$ is additive (the composition map is bilinear). This implies that

$$\mathrm{Hom}_{\mathcal{A}}\left(-, G\left(-\right)\right) \cong \mathrm{Hom}_{\mathcal{B}}\left(F\left(-\right), -\right) : \mathcal{A}^{op} \times \mathcal{B} \to \mathbf{Ab}$$

as abelian groups. Since $\mathbf{Fun}_{\mathrm{Add}}\left(\mathcal{A}, \mathcal{B}\right)$ is strictly full, $\mathrm{Hom}_{\mathcal{A}}\left(-, G\left(-\right)\right) \in \mathbf{Fun}_{\mathrm{Add}}\left(\mathcal{A}^{op} \times \mathcal{B}, \mathbf{Ab}\right) \cong \mathbf{Fun}_{\mathrm{Add}}\left(\mathcal{A}^{op}, \mathbf{Fun}_{\mathrm{Add}}\left(\mathcal{B}, \mathbf{Ab}\right)\right)$, so G is additive. $\square$

Non-additive bimodules. Let R be an associative ring with 1 (we work over $\mathbb{Z}$).

Recall that a bimodule over R is an abelian group which is both a left and right R-module and the two structures are balanced in the sense $(a \cdot m) \cdot b = a \cdot (m \cdot b), \forall m \in M, \forall a, b \in R$. Equivalently, we have the following commuting diagram.

$$\begin{tikzcd}
R \otimes M \otimes R \arrow[r, "1 \otimes m_R"] \arrow[d, "m_L \otimes 1"'] & R \otimes M \arrow[d, "m_L"] \\
M \otimes R \arrow[r, "m_R"] & M
\end{tikzcd}$$

Consider the full subcategory $\mathbf{F}\left(R\right)$ of $R\mathbf{-Mod}$ with $Ob\left(F\left(R\right)\right) = \left\{0, R, R^{\oplus 2}, \cdots, R^{\oplus n}, \cdots\right\}_{n \geq 0} \cong \mathbb{N}$. Note that $\mathbf{F}\left(R\right)$ is a small additive subcategory of $R - \mathbf{Mod}$. Consider

$$\mathcal{F}\left(R\right) := \mathbf{Fun}\left(\mathbf{F}\left(R\right), R - \mathbf{Mod}\right)$$

this is an abelian category because $\mathbf{F}\left(R\right)$ is small. (Proof see next lecture.)

There is a natural functor

$$\begin{array}{rcl} \Theta: \mathbf{Bimod}(R) & \hookrightarrow & \mathcal{F}(R) \\ M & \mapsto & \left[\begin{array}{rrcl} \hat{M}: & \mathbf{F}(R) & \to & R-\mathbf{Mod} \\ & R^n & \mapsto & M\otimes_R R^n \cong M^{\otimes n} \end{array}\right] \end{array}$$

Remark 2.2. It may seem more natural to replace $\mathbf{F}(R)$ with $\mathbf{P}(R)$ the full subcategory of finitely generated projective R-modules, but for our purposes $\mathbf{F}(R)$ suffices.

Theorem 2.1. *Θ is a fully faithful functor and it essential images are precisely $\mathbf{Fun}_{Ad}(\mathbf{F}(R), R-\mathbf{Mod})$. Thus $\Theta: \mathbf{Bimod}(R) \xrightarrow{\cong} \mathbf{Fun}_{Ad}(\mathbf{F}(R), R-\mathbf{Mod})$. Bimodules over R can be viewed as additive functors $\mathbf{F}(R) \to R-\mathbf{Mod}$.*

Definition 2.4. The objects of $\mathcal{F}(R)$ are called "*nonadditive bimodules*". $\mathcal{F}(R)$ is the category of nonadditive bimodules.

This is a nice abelian category we can do homological algebra.

Proof. The main thing we need to prove is that all additive functors are isomorphic to some $\hat{M}$. Suppose $T \in Ob(\mathcal{F}(R))$ is an additive functor. Define $M = T(R)$ the value of T on R. By definition, $M \in Ob(R-\mathbf{Mod})$. There is a natural ring isomorphism $\lambda: R^{op} \to \mathrm{End}_R(M)$ defined by

$$\begin{array}{rcccl} R^{op} & \cong & \mathrm{End}_R({}_RR) & = & \mathrm{Hom}_{\mathbf{F}(R)}(R,R) \xrightarrow{T} \mathrm{Hom}_{\mathbf{F}(R)}(TR,TR) = \mathrm{End}_R(M) \\ x & \mapsto & \left(\begin{array}{rccc} \hat{x}: & R & \to & R \\ & r & \mapsto & r\cdot x \end{array}\right) & & \end{array}$$

T is an additive functor implies that λ is a ring homomorphism. Then define

$$\begin{array}{rcl} M\otimes R & \to & M \\ m\otimes b & \mapsto & m\cdot b \coloneqq \lambda_b(m) \end{array}$$

we have

$$(a\cdot m)\cdot b = \lambda_b(a\cdot m) = a\lambda_b(m) = a\cdot(m\cdot b).$$

So M is an R-bimodule. $\square$

Exercise 2.11. The functor $\Theta : \mathbf{Bimod}(R) \hookrightarrow \mathcal{F}(R)$ has both left and right adjoints.

Question: What is the use of it?

The classical Hochschild homology can be defined by $HH^*(R, M) = \mathrm{Ext}^*_{\mathbf{Bimod}(R)}(R, M)$ where $\Theta : R \mapsto \hat{R}, M \mapsto \hat{M}$. And $THH^*(R, M) = \mathrm{Ext}^*_{\mathcal{F}(R)}\left(\hat{R}, \hat{M}\right)$.

Theorem 2.2. *(Pirashvili-Waldhauser) $THH^*(R, M)$ is canonically isomorphic to the topological Hochschild homology.*

If R is an algebra over field k then $HH^* \cong THH^*$.

2.3 Kernels and Cokernels in Additive Categories

Kernels

Let $\mathcal{A}$ be an additive category. $F, G : \mathcal{A} \to \mathbf{Ab}$ are additive functors and $\alpha : F \Rightarrow G$ is a morphism of functors. Then we can define

$$\begin{array}{rccc} \mathrm{Ker}(\alpha) : & \mathcal{A} & \to & \mathbf{Ab} \\ & Z & \mapsto & \mathrm{Ker}_{\mathbf{Ab}}\left(FZ \xrightarrow{\alpha_Z} GZ\right) \end{array}$$

This is an additive functor, which follows from the commutative diagram in $\mathbf{Ab}$

$$\begin{tikzcd} \mathrm{Ker}(\alpha)(Z) \arrow[r, hook] \arrow[d, dashed, "\mathrm{Ker}(\alpha)(\varphi)"] & FZ \arrow[r, "\alpha_Z"] \arrow[d, "F\varphi"] & GZ \arrow[d, "G\varphi"] \\ \mathrm{Ker}(\alpha)(Z') \arrow[r, hook] & FZ' \arrow[r, "\alpha_{Z'}"] & GZ' \end{tikzcd}$$

for any $\varphi : Z \to Z'$. There exists a unique arrow $\mathrm{Ker}(\alpha)(\varphi) : \mathrm{Ker}(\alpha)(Z) \to \mathrm{Ker}(\alpha)(Z')$, which defines $\mathrm{Ker}(\alpha)$ on morphisms.

We can use this to define kernels and cokernels in $\mathcal{A}$.

Let $f : X \to Y$ be any morphism in $\mathcal{A}$. We know that

$$\begin{array}{ccc} \mathcal{A} & \hookrightarrow & \mathbf{Fun}_{\mathrm{Add}}\left(\mathcal{A}^{op}, \mathbf{Ab}\right) \\ X & \mapsto & h_X := \mathrm{Hom}_{\mathcal{A}}\left(-, X\right) \\ (f : X \to Y) & \mapsto & \left(\begin{array}{cccc} h_f : & h_X & \to & h_Y \\ & g & \mapsto & f \circ g \end{array}\right) \end{array}$$

gives an embedding of additive categories, so we can associate to f a morphism of Yoneda functors $h_f : h_X \to h_Y$ and define

Definition 2.5. The *kernel* of $f : X \to Y$ in $\mathcal{A}$ is an object $K \in Ob\left(\mathcal{A}\right)$ that represents

$$\begin{array}{cccc} \mathrm{Ker}\left(h_f\right) : & \mathcal{A}^{op} & \to & \mathbf{Ab} \\ & Z & \mapsto & \mathrm{Ker}_{\mathbf{Ab}}\left(\mathrm{Hom}_{\mathcal{A}}\left(Z, X\right) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}\left(Z, Y\right)\right) \end{array}$$

By definition, if K exists, then we have a canonical exact sequence of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{A}}\left(Z, K\right) \to \mathrm{Hom}_{\mathcal{A}}\left(Z, X\right) \to \mathrm{Hom}_{\mathcal{A}}\left(Z, Y\right)$$

for any $Z \in Ob\left(\mathcal{A}\right)$. Take $Z = K$, the image of Id_K in $\mathrm{Hom}_{\mathcal{A}}\left(K, X\right)$ gives a canonical morphism $k : K \to X$. Define $\mathrm{Ker}\left(f\right) = \left(K, k\right)$ as the *kernel* of f. This is unique up to isomorphism.

Exercise 2.12. Show that for $f : X \to Y$, there is a canonical isomorphism $\mathrm{Ker}\left(f\right) \cong \mathrm{eq}\left\{ \begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & Y \end{tikzcd} \right\}$ provided both exists.

Proof. For any $g : Z \to X$ such that

$$\begin{tikzcd} Z \arrow[r, "g"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & Y \end{tikzcd}$$

we have $f \circ g = 0 : Z \to Y$, so by the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}\left(Z, K\right) \xrightarrow{k_*} \mathrm{Hom}_{\mathcal{A}}\left(Z, X\right) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}\left(Z, Y\right)$$

there must exists a unique $\bar{g} : Z \to K$ such that $k \circ \bar{g} = g$, so by the universal property of equalizer,

$\mathrm{Ker}(f) \cong \mathrm{eq}\left\{ X \underset{0}{\overset{f}{\rightrightarrows}} Y \right\}$. □

Cokernels

A similar definition of Cokernel does not work, because the functor

$$\begin{array}{rcl} \mathrm{Coker}(h_f): \ \mathcal{A}^{op} & \to & \mathbf{Ab} \\ Z & \mapsto & \mathrm{Coker}_{\mathbf{Ab}}\left(\mathrm{Hom}_{\mathcal{A}}(Z,X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(Z,Y)\right) \end{array}$$

is not representable even in simplest cases.

If $\mathrm{Coker}(h_f)$ is representable, we would have short exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(Z,X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(Z,Y) \xrightarrow{c_*} \mathrm{Hom}_{\mathcal{A}}(Z,C) \to 0$$

for any $Z \in Ob(\mathcal{A})$. This is wrong!

Counterexample: Let $\mathcal{A} = \mathbf{Ab}$, $X = Y = \mathbb{Z}$, $f : \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z}, n \geq 2$.

$$\begin{array}{ccccccc} \mathrm{Hom}(\mathbb{Z}/n,\mathbb{Z}) & \to & \mathrm{Hom}(\mathbb{Z}/n,\mathbb{Z}) & \to & \mathrm{Hom}(\mathbb{Z}/n,\mathbb{Z}/n) & \to & 0 \\ \| & & \| & & \nparallel & & \\ 0 & & 0 & & 0 & & \end{array}$$

To define $\mathrm{Coker}(f)$ in a correct way, we dualize the Yoneda embedding.

Definition 2.6. The cokernel of $f : X \to Y$ in $\mathcal{A}$ is an object $C \in Ob(\mathcal{A})$ representing

$$\begin{array}{rcl} \mathrm{Ker}\left(h^f\right): \ \mathcal{A}^{op} & \to & \mathbf{Ab} \\ Z & \mapsto & \mathrm{Ker}_{\mathbf{Ab}}\left(\mathrm{Hom}_{\mathcal{A}}(Y,Z) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(X,Z)\right) \end{array}$$

By definition, if C exists, then we have a canonical exact sequence of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{A}}(C,Z) \to \mathrm{Hom}_{\mathcal{A}}(Y,Z) \to \mathrm{Hom}_{\mathcal{A}}(X,Z)$$

for any $Z \in Ob(\mathcal{A})$. Take $Z = C$, the image of Id_C in $\mathrm{Hom}_{\mathcal{A}}(Y,C)$ gives a canonical morphism $c : Y \to C$. Define $\mathrm{Coker}(f) = (C,c)$ as the *cokernel* of f. This is unique up to isomorphism.

Exercise 2.13. Show that for $f : X \to Y$, there is a canonical isomorphism $\operatorname{Coker}(f) \cong \operatorname{coeq}\left\{ X \underset{0}{\overset{f}{\rightrightarrows}} Y \right\}$ provided both exists.

Now we have $K \xrightarrow{k} X \xrightarrow{f} Y \xrightarrow{c} C$.

Definition 2.7. For $f : X \to Y$, we define

$$\operatorname{Im}(f) = \left(\operatorname{Ker}\left(Y \xrightarrow{c} C\right), j\right)$$

$$\operatorname{Coim}(f) = \left(\operatorname{Coker}\left(K \xrightarrow{k} X\right), i\right)$$

such that

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "i"] \arrow[rrr, bend right, "f"'] & \operatorname{Coim}(f) \arrow[r, "\bar{f}"] & \operatorname{Im}(f) \arrow[r, "j"] & Y \arrow[r, "c"] & C \end{tikzcd}$$

Proposition 2.1. *Assume that $\mathcal{A}$ has kernels and cokernels, then for any $f : X \to Y$, there exists a unique $\bar{f}$: Coim $(f) \to$ Im (f) such that $f = j \circ \bar{f} \circ i$.*

Proof. Based on (repeated use of) the following lemma. Since $c \circ f = 0$, there exists a unique $f' : X \to \operatorname{Im}(f)$ such that $j \circ f' = f$. Also we have $f \circ k = j \circ f' \circ k = 0$, and use the fact that j is a monomorphism (see remark 3.1), we have $f' \circ k = 0$, so there exists a unique map $\bar{f} : \operatorname{Coim}(f) \to \operatorname{Im}(f)$ such that $f' = \bar{f} \circ i$. $\square$

Lemma 2.4. *For any $f : X \to Y$ in $\mathcal{A}$,*

1. *If Ker (f) exists, then for $g : X' \to X$, $f \circ g = 0 \iff \exists! g' : X' \to K$ such that $g = k \circ g'$.*
2. *If Coker (f) exists, then for $h : Y \to Y'$, $h \circ f = 0 \iff \exists! h' : C \to Y'$ such that $h = h' \circ c$.*

Proof. The first statement follows from the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}\left(X', K\right) \xrightarrow{k_*} \operatorname{Hom}_{\mathcal{A}}\left(X', X\right) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{A}}\left(X', Y\right)$$

$f \circ g = 0 \iff g \in \operatorname{Ker}(f_*) = \operatorname{Im}(k_*) \iff \exists! g' : X' \to K$ such that $g = k \circ g'$.

The second statement follows from the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}\left(C, Y'\right) \xrightarrow{c^*} \operatorname{Hom}_{\mathcal{A}}\left(Y, Y'\right) \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{A}}\left(X, Y'\right)$$

$h \circ f = 0 \Longleftrightarrow f \in \mathrm{Ker}\,(f^*) = \mathrm{Im}\,(c^*) \Longleftrightarrow \exists! h' : C \to Y'$ such that $h = h' \circ c$. $\square$

Example 2.4. In $\mathcal{A} = \mathbf{Ab}$ or $R - \mathbf{Mod}$, the existence of $\bar{f}$ and the fact that $\bar{f}$ is an isomorphism is part of the first isomorphism theorem.

3 Abelian Categories

Recall that we define an additive category $\mathcal{A}$ as characterized by 2.1, 2.2, 2.2. We will add one more axiom 3.1 to define abelian categories.

3.1 Abelian Categories

Definition 3.1. $\mathcal{A}$ is an *abelian category* if it is additive and also satisfying

AB4 Every $f : X \to Y$ has kernel and cokernel for any $X, Y \in Ob(\mathcal{A})$, and the canonical map $\bar{f} : \mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphsim, i.e. there exists factorization

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "i"] \arrow[rr, bend right, "f"'] & I \arrow[r, "j"] & Y \arrow[r, "c"] & C \end{tikzcd}$$

where $(K, k) = \mathrm{Ker}(f)$, $(C, c) = \mathrm{Coker}(f)$, $(I, i) = \mathrm{Coker}(k)$, $(I, j) = \mathrm{Ker}(c)$.

Example 3.1. Additive but not abelian categories.

1. $\mathbf{F}(R)/\mathbf{P}(R)$ is additive but not abelian.

2. filtered modules or abelian groups are not abelian categories.

3. topological modules or abelian groups are not abelian.

Definition 3.2. Let $f : X \to Y$ be a morphism in $\mathcal{A}$.

1. f is *mono* if $K = 0$, and we denote $f : X \hookrightarrow Y$. In this case, X is called a subobject of Y.

2. f is epi if $C = 0$, and we denote $f : X \twoheadrightarrow Y$. In this case, Y is called the quotient of X.

Exercise 3.1. Show that for any $f : X \to Y$, k is mono and c is epi.

Proof. Let $(K', k') = \mathrm{Ker}(K)$, we have

$$K' \xrightarrow{k'} K \xrightarrow{k} X \xrightarrow{f} Y$$

such that $f \circ k = 0$ and $k \circ k' = 0$. By the universal property we know that $k' = 0$ (unique factorization), so $K = 0$. k is mono and similarly we have c is epi. □

Lemma 3.1. *(Grothendieck) The axiom 3.1 is equivalent to the combination of two axioms,*

AB4.1 *$\mathcal{A}$ has kernels and cokernels.*

AB4.2 *If for $f : X \to Y$ we have $Ker(f) = Coker(f) = 0$, then f is an isomorphism (i.e. f is both mono and epi, see definition below).*

In particular, this says that kernel and cokernel exists for every $f \in \mathrm{Mor}(\mathcal{A})$.

Proof. Assume that the axiom 3.1 holds, given $f : X \to Y$, we can define $(K, k) = \mathrm{Ker}(f), (C, c) = \mathrm{Coker}(f), (I, i) = \mathrm{Coker}(k), (I', j) = \mathrm{Ker}(c)$. And we have a factorization as follows

$$\begin{tikzcd} K \arrow[r, "k"] & X \arrow[r, "i"] \arrow[rrr, bend right, "f"'] & I \arrow[r, "\bar{f}"] & I' \arrow[r, "j"] & Y \arrow[r, "c"] & C \end{tikzcd}$$

Then $\mathrm{Coker}\left(\bar{f}\right) = \mathrm{Ker}\left(\bar{f}\right) = 0$ (Why?). By the axiom 3.1 $\bar{f} : I \to I'$ is an isomorphism.

Assume that the axiom 3.1 holds, Consider $f : X \to Y$ with $\mathrm{Ker}(f) = \mathrm{Coker}(f) = 0$, then we have an isomorphism

$$(X, \mathrm{Id}_X) = \mathrm{Coker}(0 \to X) \cong \mathrm{Ker}(Y \to 0) = (Y, \mathrm{Id}_Y)$$

which is f itself.

$$\begin{tikzcd} 0 \arrow[r, "k"] & X \arrow[r, "\mathrm{Id}_X"] \arrow[rrr, bend right, "f"'] & X \arrow[r, "f"] & Y \arrow[r, "\mathrm{Id}_Y"] & Y \arrow[r, "c"] & 0 \end{tikzcd}$$

Hence f is an isomorphism. □

Exercise 3.2. In any abelian category, for any composable $f : X \to Y$ and $g : Y \to Z$, we have

1. $\mathrm{Ker}(f) \hookrightarrow \mathrm{Ker}(gf)$ and they are equal if $\mathrm{Ker}(g) = 0$.

2. $\mathrm{Coker}(gf) \hookrightarrow \mathrm{Coker}(g)$ and they are equal if $\mathrm{Coker}(f) = 0$.

3. If f is epi then $gf = 0 \iff g = 0$. If g is mono then $gf = 0 \Longleftarrow f = 0$.

Remark 3.1. f is mono is equivalent to say that for any $g, h : Z \to X$, if $f \circ g = f \circ h$, then $g = h$. This is because $f \circ (g - h) = f \circ g - f \circ h = 0$, and by definition, $g - h$ factors through $K = 0$, so

$g-h=0$, i.e. $g=h$. f is epi is is equivalent to say that for any $g,h:Y\to Z$, if $g\circ f=h\circ f$, then $g=h$. This is because $(g-h)\circ f=g\circ f-h\circ f=0$, and by definition, $g-h$ factors through $C=0$, so $g-h=0$, i.e. $g=h$.

3.2 Examples of Abelian Categories

The categories **Ab**, $R-\mathbf{Mod}$, $\mathbf{Mod}-R$ are abelian, where R is an associative unital ring. The category of R-bimodules

$$\begin{array}{ccccc} \mathbf{Bimod}(R) & \cong & \mathbf{Mod}(R^e) & \cong & R^e-\mathbf{Mod} \\ a\cdot m\cdot b & = & m\cdot(b\otimes a^o) & = & (a\otimes b^o)\cdot m \end{array}$$

where $R^e=R\otimes R^{op}$ is also abelian.

In these example, kernels and cokernels are defined in the usual way and the axiom 3.1 is equivalent to the first isomorphism theorem.

Lemma 3.2. *We have*

1. *$\mathcal{A}$ is abelian if and only if $\mathcal{A}^{op}$ is abelian. (All axioms are self-dual).*

2. *If $\mathcal{A}$, $\mathcal{B}$ are abelian, so is $\mathcal{A}\times\mathcal{B}$.*

Lemma 3.3. *If $\mathcal{C}$ is small and $\mathcal{A}$ is abelian, then* $\mathbf{Fun}(\mathcal{C},\mathcal{A})$ *is abelian.*

Proof. We've shown that if $\mathcal{A}$ is additive, $\mathcal{A}^{\mathcal{C}}$ is additive. We only need to check the axiom 3.1. Kernels and cokernels in $\mathcal{A}^{\mathcal{C}}$ are defined pointwisely. Let $F,G:\mathcal{C}\to\mathcal{A}$ be two functors, $\varphi:F\Rightarrow G$ a morphism of functors. Define $\mathrm{Ker}(\varphi)=(K:\mathcal{C}\to\mathcal{A},k:K\Rightarrow F)$ as follows:

K is given by:

Objects: $\forall X\in Ob(\mathcal{C})$, $KX=\mathrm{Ker}\left(FX\xrightarrow{\varphi_X}GX\right)$. Note that kernels in $\mathcal{A}$ induces morphisms $k_X:KX\to FX,\forall X\in Ob(\mathcal{C})$.

Morphisms: $\forall f:X\to Y$ in $\mathcal{C}$, we have

$$\begin{tikzcd} KX \arrow[r, "k_X"] \arrow[d, dashed, "Kf"] & FX \arrow[r, "\varphi_X"] \arrow[d, "Ff"] & GX \arrow[d, "Gf"'] \\ KY \arrow[r, "k_Y"] & FY \arrow[r, "\varphi_Y"] & GY \end{tikzcd}$$

where there exists a unique $Kf : KX \to KY$ which completes the (commutative) diagram. Indeed, $\varphi_Y \circ Ff \circ k_X = Gf \circ \varphi_X \circ k_X = 0$, so $Ff \circ k_X$ factors through k_Y, which gives $Kf : KX \to KY$. This defines $K(f)$ and shows that $k = \{k_X : KX \to GX\}_{X \in Ob(\mathcal{C})}$ is a morphism of functors. (K, k) is the kernel, since it satisfies the universal property by checking it pointwise and satisfies naturality.

Coker $(\varphi) = (C, c)$ is defined similarly "pointwise" or "objectwise".

The axiom 3.1 holds because it holds in $\mathcal{A}$. Coker $(k) \xrightarrow{\bar{\varphi}}$ Ker (c) is given by

$$\bar{\varphi} = \left\{ \bar{\varphi}_X : \mathrm{Coker}\left(\mathrm{Ker}\left(\varphi_X\right)\right) \xrightarrow{\cong} \mathrm{Ker}\left(\mathrm{Coker}\left(\varphi_X\right)\right) \right\}_{X \in Ob(\mathcal{C})}$$

so $\bar{\varphi}$ is an isomorphism in$\mathcal{A}^{\mathcal{C}}$. $\square$

Corollary 3.1. *The category of nonadditive bimodules $\mathcal{F}(R)$ is abelian.*

Corollary 3.2. *If X is a topological space, the category* $\mathbf{Pr}(X)$ *of presheaves on abelian categories is abelian.*

Question: what about sheaves?

3.3 Sheaves of Abelian Groups

Let X be a topological space. A *presheaf* is a functor

$$\begin{array}{rccc} F: & \mathbf{Open}(X)^{op} & \to & \mathbf{Ab} \\ & U & \mapsto & FU \end{array}$$

with

$$\mathrm{Hom}(U, V) = \begin{cases} \emptyset & U \nsubseteq V \\ \to & U \subseteq U \end{cases}$$

and $F(U \to V) = \rho_U^V : FV \to FU$ called the restriction map, satisfying $\forall U \subseteq V \subseteq W$, $\rho_U^V \circ \rho_V^W = \rho_U^W$, i.e. F preserves compositions.

Definition 3.3. (Sheaf Axiom) A presheaf is a *sheaf* if for any open subset $U \subset X$ and open cover $\{U_i\}_{i \in I}$ of U, and any sections $\{e_i \in FU_i\}_{i \in I}$ such that $\rho^{U_i}_{U_i \cap U_j}(e_i) = \rho^{U_i}_{U_i \cap U_i}(e_j)$ whenever $U_i \cap U_j \neq \emptyset$, then we have a unique $e \in FU$ such that $e_i = \rho^U_{U_i}(e), \forall i \in I$.

Note that a presheaf fails to be a sheaf if either there are not enough global sections or uniqueness of restrictions fails.

Definition 3.4. $\mathbf{Sh}(X)$ is the full subcategory of $\mathbf{Pr}(X)$ so it comes together with an inclusion functor

$$s : \mathbf{Pr}(X) \rightleftarrows \mathbf{Sh}(X) : i$$

Theorem 3.1. *This inclusion functor* $i : \mathbf{Sh}(X) \to \mathbf{Pr}(X)$ *has a left adjoint* $s : \mathbf{Pr}(X) \to \mathbf{Sh}(X)$.

Remark 3.2. We are trying to make $\mathbf{Sh}(X)$ abelian in naive way., i.e. for $\varphi : F \Rightarrow G$ a morphism of sheaves, define

$$\mathrm{Ker}_{\mathbf{Sh}}(\varphi) = \mathrm{Ker}_{\mathbf{Pr}}(i\varphi) = \{\mathrm{Ker}(\varphi_U : FU \to GU)\}_{U \in Ob(\mathbf{Open}(X))}$$

$$U \mapsto \mathrm{Ker}(\varphi)(U) = \mathrm{Ker}(\varphi_U : FU \to GU) = \mathrm{Ker}_{\mathbf{Pr}}(i\varphi)(U)$$

$$U \mapsto \mathrm{Coker}(\varphi)(U) = \mathrm{Coker}(\varphi_U : FU \to GU) = \mathrm{Coker}_{\mathbf{Pr}}(i\varphi)(U)$$

Exercise 3.3. Check that $\mathrm{Ker}(\varphi)$ is a sheaf by $\mathrm{Coker}(\varphi)$ is not a sheaf.

The correct way to define $\mathrm{Coker}(\varphi)$ is

Definition 3.5. Define $U \mapsto \mathrm{Coker}(\varphi)(U)$ by $f \in \mathrm{Coker}(\varphi)(U) \iff$ there exists a covering $\{U_i\}_{i \in I}$ of U such that $\rho^U_{U_i}(f) \in \mathrm{Coker}_{\mathbf{Pr}}(i\varphi)(U)$. This gives a sheaf which is called $\mathrm{Coker}(\varphi)$.

Assume that the axiom is true, then we have

$$s : \mathbf{Pr}(X) \rightleftarrows \mathbf{Sh}(X) : i$$

such that $si \cong \mathrm{Id}_{\mathbf{Sh}(X)}$.

Proposition 3.1. *If* $\varphi : F \Rightarrow G$ *in* $\mathbf{Sh}(X)$, *consider the decomposition*

$$K \xrightarrow{k} iF \xrightarrow{i} I \xrightarrow{j} iG \xrightarrow{c} C$$

in $\mathbf{Pr}(X)$ *, apply* s *to this we have*

$$K \cong sK \xrightarrow{sk} F \xrightarrow{si} sI \xrightarrow{sj} G \xrightarrow{sc} sC \neq C$$

This is a decomposition in $\mathbf{Sh}(X)$*.*

This holds (in general) abstractly.

Proposition 3.2. *Given an adjoint pair* $s : \mathcal{A} \leftrightarrows \mathcal{B} : i$ *such that* $s \circ i \cong Id_{\mathcal{B}}$ *where* $\mathcal{A}$ *is abelian, we can make* $\mathcal{B}$ *abelian by defining*

$$Ker_{\mathcal{B}}\varphi := Ker_{\mathcal{A}}(i\varphi)$$
$$Coker_{\mathcal{B}}\varphi := s(Coker_{\mathcal{A}}(i\varphi))$$

This is called transfer principle.

There is a similar statement for model/triangulated categories.

4 Homological Algebra

4.1 Complexes

Let $\mathcal{A}$ be an abelian category.

Definition 4.1. A *cochain (cohomological) complex* in $\mathcal{A}$ is a sequence of morphisms $X^\bullet = (X^n, d^n)_{n\in\mathbb{Z}}$

$$\cdots \to X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \to \cdots$$

satisfying that $d^2 = 0$.

Definition 4.2. Dually, a *chain (homological) complex* in $\mathcal{A}$ is a sequence of morphisms $X_\bullet = (X_n, d_n)_{n\in\mathbb{Z}}$

$$\cdots \to X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \to \cdots$$

satisfying that $d^2 = 0$.

Convention. When working with unbounded complexes (no extra constraints) we can pass from cohomological to homological notation by setting $X_n^{\text{hom}} := X^{-n}, \forall n \in \mathbb{Z}$.

Any property or fact for cohomological complexes has an analogue for homological complexes. By a complex, we will mean cohomological complex (unless stated otherwise).

Definition 4.3. A morphism of complexes $f^\bullet : X^\bullet \to Y^\bullet$ is a sequence of morphisms $(f^n : X^n \to Y^n)_{n\in\mathbb{Z}}$ such that

$$\begin{tikzcd}
\cdots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes, for any $n \in \mathbb{Z}$. Write $\mathbf{Com}(\mathcal{A})$ for the category of (unbounded) complexes of $\mathcal{A}$.

Theorem 4.1. $\mathbf{Com}(\mathcal{A})$ *is an abelian category with kernels and cokernels defined term-wise (or degree-wise).*

Proof is similar to that $\mathbf{Fun}(\mathcal{C}, \mathcal{A})$ is abelian, where $\mathcal{C}$ is small.

Example 4.1. If $\mathcal{A} = R - \mathbf{Mod}$, $\mathbf{Com}(R - \mathbf{Mod})$ is the category of complexes of R-modules.

Example 4.2. (DG-modules) Let R be a differential $\mathbb{Z}$-graded ring $R = \bigoplus_{n\in\mathbb{Z}} R^n$ as a $\mathbb{Z}$-module, i.e.

1. $R^n \cdot R^m \subseteq R^{n+m}$
2. $1_R \in R^0$

such that R is equipped with differential d

$$\cdots \to R^n \xrightarrow{d^n} R^{n+1} \xrightarrow{d^{n+1}} R^{n+2} \to \cdots$$

such that $d^{n+1}d^n = 0$, making it a complex, and d satisfies the Leibniz rule, i.e. for homogeneous $a, b \in R$,

$$d(ab) = (da) \cdot b + (-1)^n a \cdot (db), \quad a \in R^n \ \cdot$$

Note that $d(1_R) = 0$ by the rule.

Example 4.3. An ordinary ring R can be viewed as a DG ring by setting $R^\bullet = [0 \to R \to 0]$ with $d = 0$. **Rings** $\hookrightarrow$ **DGRings** is a fully faithful embedding.

Remark 4.1. DG rings are monoids in the monoidal category $\mathbf{Com}(\mathbf{Ab})$ with respect to tensor product

$$\begin{array}{rccc} \bigotimes: & \mathbf{Com}(\mathbf{Ab}) \times \mathbf{Com}(\mathbf{Ab}) & \to & \mathbf{Com}(\mathbf{Ab}) \\ & (X^\bullet, Y^\bullet) & \mapsto & X^\bullet \bigotimes Y^\bullet \end{array}$$

where $(X \bigotimes Y)^n = \bigoplus_{i+j=n} X^i \otimes Y^j$ and $d : (X \bigotimes Y)^n \to (X \bigotimes Y)^{n+1}$ is given by $d_{X \bigotimes Y} = \mathrm{Id}_X \otimes d_Y + d_X \otimes \mathrm{Id}_Y$, or explicitly, for $x \in X^n$ and $y \in Y^m$, $d(x \otimes y) = dx \otimes y + (-1)^n x \otimes dy$.

Definition 4.4. A left DG module $M^\bullet$ over a DG ring $R^\bullet$ is a graded $R^\bullet$-module such that

1. $M^\bullet = \bigoplus_{n\in\mathbb{Z}} M^n$ with left $R^\bullet$-action $R^\bullet \otimes M^\bullet \to M^\bullet$ such that $R^n \otimes M^k \to M^{n+k}, a \otimes m \mapsto a \cdot m$.
2. there is a differential d_M on $M^\bullet$ making $(M^\bullet, d_M)$ a complex.
3. $d_M(a \cdot m) = d_R(a) \cdot m + (-1)^{|a|} a \cdot d_M(m), \forall a \in R^{|a}, m \in M$.

Write $\mathbf{DGMod}(R)$ for the category of DG modules over R.

Notice that if we think of a ring R as a DG ring, then $\mathbf{Com}(R - \mathbf{Mod}) = \mathbf{DGMod}(R)$ with $|a| = 0, d_R = 0$.

Exercise 4.1. **DGMod** (R) is an abelian category.

Example 4.4. (Mixed complexes) Take $\Lambda = \Lambda(x) = k[x]/(x^2) = \underset{(0)}{k} \oplus \underset{(1)}{kx}$ with $|x| = 1$ and $d_R = 0$.

Definition 4.5. A right homological DG module over Λ is called a *mixed complex*. Explicitly, a mixed complex (M, b, B) is a complex $M_\bullet = \left[\cdots \to M_{n+1} \xrightarrow{b_{n+1}} M_n \xrightarrow{b_n} M_{n-1} \to \cdots\right]$

1. with $b^2 = 0, \{b\} = -1$, and

2. a right Λ-action $M \otimes \Lambda \to M$, i.e.

$$\begin{aligned} B: \quad M_n &\to \quad M_{n+1} \\ x &\mapsto (-1)^n m \cdot x \end{aligned}$$

   where $B^2 = 0, |B| = 1$.

3. it satisfies Leibniz rule, $bB + Bb = 0$.

Cohomology

Fix a complex $X^\bullet = (X^n, d^n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$ and consider

$$\begin{tikzcd}
 & & \operatorname{Coker}(d^n) \arrow[dr, dashed, "b^{n+1}"] & & \\
\cdots \arrow[r] & X^n \arrow[r, "d^n"] \arrow[dr, dashed, "a^n"'] & X^{n+1} \arrow[u, "c"] \arrow[r, "d^{n+1}"] & X^{n+2} \arrow[r] & \cdots \\
 & & \operatorname{Ker}(d^{n+1}) \arrow[u, "k"'] & &
\end{tikzcd}$$

since $d^{n+1}d^n = 0$, there exists a unique $a^n : X^n \to \operatorname{Ker}(d^{n+1})$ such that $k \circ a^n = d^n$ and a unique $b^{n+1} : \operatorname{Coker}(d^n) \to X^{n+2}$ such that $d^{n+1} = b^{n+1} \circ c$.

By the axiom 3.1 we have

$$\begin{tikzcd}
X^n \arrow[r, "a^n"] \arrow[d, dashed, "\bar{a}^n"] & \operatorname{Ker}(d^{n+1}) \arrow[r, hook, "k"] \arrow[d, dashed] & X^{n+1} \arrow[r, two heads, "c"] & \operatorname{Coker}(d^n) \arrow[r, "b^{n+1}"] \arrow[dr, "c'"] & X^{n+2} \\
K \arrow[ur, "k'"] & \operatorname{Coker}(a^n) \arrow[rr, dashed, "\overline{cok}", "\cong"'] & & \operatorname{Ker}(b^{n+1}) \arrow[u, dashed] & C \arrow[u, dashed, "\bar{b}^{n+1}"']
\end{tikzcd}$$

note that $\operatorname{Coker}(a^n) \cong \operatorname{Coker}(k')$ and $\operatorname{Ker}(b^{n+1}) \cong \operatorname{Ker}(c')$.

Definition 4.6. The *n-th cohomology* $H^n(X)$ is defined by $H^n(X) := \operatorname{Coker}(a^n) \cong \operatorname{Ker}(b^{n+1})$.

This defines additive functors $H^n : \mathbf{Com}(\mathcal{A}) \to \mathcal{A}$. We will often denote

$$\begin{aligned} H^\bullet : \quad \mathbf{Com}(\mathcal{A}) &\to \quad \mathbf{Com}(\mathcal{A}) \\ (X^\bullet, d) &\mapsto \quad (H^\bullet(X), 0) \end{aligned}$$

Exercise 4.2. If R is a DG ring, M is a DG module, then $H^\bullet(M)$ is a graded $H^\bullet(R)$-module, so $H^\bullet$ restricts to a functor $H^\bullet : \mathbf{DGMod}(R) \to \mathbf{grMod}(H^\bullet(R))$.

Proof. Let $[x] \in H^n(R) = \mathrm{Ker} d_R^{n+1} / \mathrm{Im} d_R^n$ and $[m] \in H^l(M) = \mathrm{Ker} d_M^{l+1} / \mathrm{Im} d_M^l$, then $[x] \cdot [m] = (x + \mathrm{Im} d_R^n) \cdot (m + \mathrm{Im} d_M^l) = xm + \mathrm{Im} d_M^{n+l+1}$ since $\forall m' \in M^l, \forall x' \in R^n$, and $\forall x \in \mathrm{Ker} d_R^{n+1}, \forall m \in \mathrm{Ker} d_M^{l+1}$, we have $x \cdot dm' = d(xm') \in \mathrm{Im} d_M^{n+l+1}$ and $dx' \cdot m = d(x'm) \in \mathrm{Im} d_M^{n+l+1}$, and $dx' \cdot dm' = d(x' \cdot dm') \in \mathrm{Im} d_M^{n+l+1}$. Also, $d\left(xm + \mathrm{Im} d_M^{n+l+1}\right) = d(xm) = dx \cdot m + (-1)^{n+1} x \cdot dm = 0$, so $[x] \cdot [m] \in \mathrm{Ker} d_M^{n+l+2} / \mathrm{Im} d_M^{n+l+1}$. $\square$

Definition 4.7. A complex $X^\bullet \in \mathbf{Com}(\mathcal{A})$ is *exact* in degree n if $H^n(X) = 0$. A complex $X^\bullet \in \mathbf{Com}(\mathcal{A})$ is *acyclic* if $H^\bullet(X) = 0$.

In classical homological algebra, the complexes

$$0 \to X \to Y \to Z \to 0 \tag{20}$$

play a distinguished role ("test functions" for additive functors).

Definition 4.8. The complex 20 is called

1. left exact if the complex 20 is exact at X and Y.

2. right exact if the complex 20 is exact at Y and Z.

3. short exact sequence if the complex 20 is exact at X, Y and Z.

Example 4.5. For any $f : X \to Y$, $0 \to \operatorname{Ker}(f) \xrightarrow{i} X \xrightarrow{f} Y \to 0$ is left exact, and $0 \to X \xrightarrow{f} Y \xrightarrow{j} \operatorname{Coker}(f) \to 0$ is right exact.

Lemma 4.1. $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is*

1. *left exact if and only if* $0 \to Hom(A, X) \xrightarrow{f_*} Hom(A, Y) \xrightarrow{g_*} Hom(A, Z) \to 0$ *is left exact for any* $A \in Ob(\mathcal{A})$.

2. *right exact if and only if* $0 \to Hom(Z, A) \xrightarrow{g^*} Hom(Y, A) \xrightarrow{f^*} Hom(X, A) \to 0$ *is left exact for any* $A \in Ob(\mathcal{A})$.

Proof. If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is left exact, f is mono and $\mathrm{Ker}(g) = \mathrm{Im}(f)$. Consider $0 \to \mathrm{Hom}(A, X) \xrightarrow{f_*} \mathrm{Hom}(A, Y) \xrightarrow{g_*} \mathrm{Hom}(A, Z) \to 0$, if $f^*(h) = f \circ h = 0$ where $h : A \to X$, since f is mono, $h = 0$, so f_* is mono. $\mathrm{Im}(f_*) \subseteq \mathrm{Ker}(g_*)$ since $g_* \circ f_*(h) = g \circ f \circ h = 0, \forall h : A \to X$. For any $t : A \to Y$ such that $g_*(t) = g \circ t = 0$, for any $a \in A$, $t(a) \in \mathrm{Ker}(g) = \mathrm{Im}(f)$, so there exists a unique $x \in X$ such that $f(x) = t(a)$, set $s : A \to X, a \mapsto x$, then $s(a_1 + a_2) = x_1 + x_2 = s(a_1) + s(a_2)$ and $s(ra) = rx = r \cdot s(a)$ since $t(ra) = rt(a) = rf(x) = f(rx)$, therefore $s \in \mathrm{Hom}(A, X)$. Hence $\mathrm{Im}(f_*) = \mathrm{Ker}(g_*)$. Thus $0 \to \mathrm{Hom}(A, X) \xrightarrow{f_*} \mathrm{Hom}(A, Y) \xrightarrow{g_*} \mathrm{Hom}(A, Z) \to 0$ is left exact for any $A \in Ob(\mathcal{A})$.

If $0 \to \mathrm{Hom}(A, X) \xrightarrow{f_*} \mathrm{Hom}(A, Y) \xrightarrow{g_*} \mathrm{Hom}(A, Z) \to 0$ is left exact for any $A \in Ob(\mathcal{A})$, since f_* is mono, i.e. for any $h : A \to X$, $f \circ h = 0$ implies $h = 0$, it follows that f is mono. Take $A = X$ and $\mathrm{Id}_X \in \mathrm{Hom}(X, X)$, $g_* \circ f_*(\mathrm{Id}_X) = g \circ f = 0$, so $\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. Take $A = \mathrm{Ker}(g)$ and $k : A \to Y$, since $g_*(k) = g \circ k = 0$, $k \in \mathrm{Im}(f_*)$, i.e. $k = f \circ h$ for some $h : A \to X$, so $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$. Hence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is left exact.

If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is right exact, g is epi and $\mathrm{Ker}(g) = \mathrm{Im}(f)$. Consider $0 \to \mathrm{Hom}(Z, A) \xrightarrow{g^*} \mathrm{Hom}(Y, A) \xrightarrow{f^*} \mathrm{Hom}(X, A) \to 0$, since g is epi, for any $h : Z \to A$, $g^*(h) = h \circ g = 0$ implies $h = 0$, so g^* is mono. $f^* \circ g^* = (g \circ f)^* = 0$, so $\mathrm{Im}(g^*) \subseteq \mathrm{Ker}(f^*)$. If $t \in \mathrm{Ker}(f^*)$, i.e. $0 = t \circ f : X \to A$, for any $z \in Z$, since g is epi, there exists $y \in Y$ such that $g(y) = z$, set $h : Z \to A, z \mapsto t(y)$. This map is well-defined, since if there exists $y' \in Y$ such that $g(y') = z$, $y - y' \in \mathrm{Ker}(g) = \mathrm{Im}(f)$, say $y - y' = f(x)$ for $x \in X$, then $t(y) - t(y') = t(y - y') = t(f(x)) = 0$. Hence $\mathrm{Im}(g^*) = \mathrm{Ker}(f^*)$, $0 \to \mathrm{Hom}(Z, A) \xrightarrow{g^*} \mathrm{Hom}(Y, A) \xrightarrow{f^*} \mathrm{Hom}(X, A) \to 0$ is left exact.

If $0 \to \mathrm{Hom}(Z, A) \xrightarrow{g^*} \mathrm{Hom}(Y, A) \xrightarrow{f^*} \mathrm{Hom}(X, A) \to 0$ is left exact, g^* is mono and $f^* \circ g^* = (g \circ f)^* = 0$, so g is epi and $g \circ f = 0$. Consider $A = \mathrm{Coker}(f)$ and $c : Y \to A$, then $f^*(c) = c \circ f = 0$, so $c \in \mathrm{Ker}(f^*) = \mathrm{Im}(g^*)$, $c = h \circ g$ for some $h : Z \to A = Y/\mathrm{Im}(f)$, so $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$. Hence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is right exact. □

4.2 Exact Functors

Let $\mathcal{A}, \mathcal{B}$ be two abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an additive functor.

Definition 4.9. F is called

1. left exact if $0 \to FX \to FY \to FZ$ is exact for any left exact triple $0 \to X \to Y \to Z$.

2. right exact if $FX \to FY \to FZ \to 0$ is exact for any right exact triple $X \to Y \to Z \to 0$.

Example 4.6. For any $A \in Ob\,(\mathcal{A})$,

1. $h^A : \mathcal{A} \to \mathbf{Ab}, X \mapsto \mathrm{Hom}\,(A, X)$ is left exact.

2. $h_A : \mathcal{A}^{op} \to \mathbf{Ab}, X \mapsto \mathrm{Hom}\,(X, A)$ is right exact.

Theorem 4.2. *(Adjointness v.s. Exactness) Let $F : \mathcal{A} \rightleftharpoons \mathcal{B} : G$ be a pair of adjoint additive functors between two abelian categories, then F is right exact and G is left exact.*

Proof. To see that F is right exact, it suffices to show that for any right exact $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$,

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}\,(FX, A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}\,(FY, A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}\,(FZ, A) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}\,(X, GA) \arrow[r] & \mathrm{Hom}\,(Y, GA) \arrow[r] & \mathrm{Hom}\,(Z, GA)
\end{tikzcd}$$

is left exact. This follows from Lemma 4.1 and the fact that h^A is left exact. □

Corollary 4.1. *If F is both left adjoint and right adjoint, then F is exact.*

Example 4.7. If $f : A \to B$ is a ring homomorphism, we have the triple of additive functors $\left(f^*, f_*, f^!\right)$, (resp. extension, restriction, coinduction)

$$\begin{tikzcd}
B - \mathbf{Mod} \arrow[d, "f_*"] \\
A - \mathbf{Mod} \arrow[u, bend left=60, "f^*"] \arrow[u, bend right=60, "f^!"']
\end{tikzcd}$$

where

$$\begin{aligned} f^*: \quad A-\mathbf{Mod} &\to B-\mathbf{Mod} \\ M &\mapsto B\otimes_A M \end{aligned}$$

when we consider B as a B,A-bimodule ${}_BB_A$, whose right A-module structure is given by $b\cdot a = b\cdot f(a)$, and

$$\begin{aligned} f^!: \quad A-\mathbf{Mod} &\to B-\mathbf{Mod} \\ M &\mapsto \mathrm{Hom}_{\mathcal{A}}({}_AB_B, M) \end{aligned}$$

Then f^* is right exact, f_* is exact and $f^!$ is left exact.

Example 4.8. (Zuckerman functor) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}\subseteq\mathfrak{g}$ a Cartan subalgebra. If $M\in Ob(\mathfrak{g}-\mathbf{Mod})$, an element $m\in M$ is called $\mathfrak{h}$-finite if $\dim_{\mathbb{C}}(\mathcal{U}\mathfrak{h}\cdot m)<\infty$. The subspace of all $\mathfrak{h}$-finite elements in M form a $\mathfrak{g}$-submodule $M_{\mathfrak{h}}=\{m\in M|\dim_{\mathbb{C}}(\mathcal{U}\mathfrak{h}\cdot m)<\infty\}$. The Zuckerman functor

$$(-)_{\mathfrak{h}}:\mathfrak{g}-\mathbf{Mod}\to\mathfrak{g}-\mathbf{Mod}$$

is left exact (not exact) but not right adjoint.

Mitchell Theorem

In $\mathbf{Ab}$ and $R-\mathbf{Mod}$, various properties (i.e. commutativity of diagrams) can be checked element-wise by diagrammatic chasing. In an arbitrary abelian category with no forgetful functor to $\mathbf{Set}$, objects do not have elements. There are two ways to deal with this.

1. Given object $X\in Ob(\mathcal{A})$ we can look at "Y-elements" or "Y-points" $\mathrm{Hom}_{\mathcal{A}}(Y,X)$ for each $Y\in Ob(\mathcal{A})$, i.e. identifying X with functors h_X via the embedding $\mathcal{A}\hookrightarrow\mathbf{Fun}(\mathcal{A}^{op},\mathbf{Ab})$.

2. Based on Mitchell Theorem.

Theorem 4.3. *(Mitchell) Let $\mathcal{A}$ be a small abelian category, then there exists an associative unital ring R together with an exact strictly fully faithful functor $A\hookrightarrow R-\mathbf{Mod}$, i.e. $\mathcal{A}$ can be identified with a full subcategory of a module category.*

In practice, any property or claim which involves finitely many objects, morphisms and which holds in any module category must hold in any abelian category $\mathcal{A}$. Indeed, we can always take $\mathcal{A}_0$ to be generated the objects involved and apply Mitchell embedding.

Example 4.9. (Snake lemma) In any abelian category $\mathcal{A}$, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\alpha"] & B_1 \arrow[r] \arrow[d, "\beta"] & C_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0
\end{tikzcd}$$

with exact rows, then there exists $\delta : \mathrm{Ker}\,(\gamma) \to \mathrm{Coker}\,(\alpha)$ making the exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & Ker(\alpha)(\mathcal{A}) \arrow[r] & Ker(\beta) \arrow[r] & H^{n+1}(\mathcal{C}) \arrow[dlll, rounded corners, "\alpha^0" description] & \\
H^n(\mathcal{A}) \arrow[r] & H^n(\mathcal{B}) \arrow[r] & H^n(\mathcal{C}) \arrow[r] & 0
\end{tikzcd}$$

It suffices to check this in module category, see Lemma 4.8.

Definition 4.10. In any category $\mathcal{A}$, a square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "p"] \\
Z \arrow[r, "q"] & W
\end{tikzcd} \tag{21}$$

is called

1. Cartesian if $X \cong Y \times_W Z$, i.e. X represents the functor

$$\begin{array}{ccc}
\mathcal{A}^{op} & \longrightarrow & \mathbf{Set} \\
A & \longmapsto & \{(\varphi, \psi) \in \mathrm{Hom}\,(A, Y) \times \mathrm{Hom}\,(A, Z)\, | p\varphi = q\psi\}
\end{array}$$

2. Cocartesian if $W \cong Y \sqcup_X Z$, i.e. W corepresents the functor

$$\begin{array}{ccc}
\mathcal{A} & \longrightarrow & \mathbf{Set} \\
A & \longmapsto & \{(\varphi, \psi) \in \mathrm{Hom}\,(Y, A) \times \mathrm{Hom}\,(Z, A)\, | \varphi f = \psi g\}
\end{array}$$

Exercise 4.3. Let $\mathcal{A}$ be an abelian category, define the sequence associated to the diagram 21

$$0 \to \underset{0}{X} \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} \underset{1}{Y \oplus Z} \xrightarrow{\begin{pmatrix} p \\ -q \end{pmatrix}} \underset{2}{W} \to 0 \tag{22}$$

Show that

1. the complex 22 is a complex if and only if the diagram 21 commutes.

2. the complex 22 is left exact if and only if the diagram 21 is Cartesian.

3. the complex 22 is right exact if and only if the diagram 21 is Cocartesian.

Proof. (1) is trivial. We will prove (2). The complex 22 is left exact if and only if X is the kernel of $Y \oplus Z \to W$, if and only if for any A and $f' : A \to Y, g' : A \to Z$ such that $g' \circ q = f' \circ p$, i.e. the composition $A \to Y \oplus Z \to W$ is zero, there exists a unique $t : A \to X$ such that

$$\begin{tikzcd}
A \arrow[drr, bend left, "f'"] \arrow[dr, dashed, "t"] \arrow[ddr, bend right, "g'"'] & & \\
& X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "p"] \\
& Z \arrow[r, "q"] & W
\end{tikzcd}$$

if and only if the diagram 21 is Cartesian.

(3) can be proved in the similar manner. The complex 22 is right exact if and only if W is the cokernel of $X \to Y \oplus Z$ if and only if the diagram 21 is Cocartesian. $\square$

Exercise 4.4. Let H^0, H^1, H^2 denote the cohomology of the complex 22. Prove or disprove

1. $H^0 \cong \mathrm{Ker}\,(g : \mathrm{Ker} f \to \mathrm{Ker} q)$.

2. $H^1 \cong \mathrm{Coker}\,(p : \mathrm{Coker} f \to \mathrm{Coker} q)$

3. There exists an exact triple

$$0 \to \mathrm{Coker}\,(g : \mathrm{Ker} f \to \mathrm{Ker} q) \to H^2 \to \mathrm{Ker}\,(p : \mathrm{Coker} f \to \mathrm{Coker} q) \to 0$$

Exercise 4.5. Show that

1. The diagram 21 is Cartesian if and only if g induces an isomorphism $\mathrm{Ker}\,(f) \cong \mathrm{Ker}\,(q)$ and p induces a monomorphism $\mathrm{Coker}\,(f) \hookrightarrow \mathrm{Coker}\,(q)$.

2. The diagram 21 is Cocartesian if and only if p induces an isomorphism $\mathrm{Coker} f \xrightarrow{\cong} \mathrm{Coker}\,(q)$ and g induces an epimorphism $\mathrm{Ker}\,(f) \twoheadrightarrow \mathrm{Ker}\,(q)$

Hint: It suffices to prove (1) and replacing $\mathcal{A}$ with $\mathcal{A}^{op}$ will imply (2).

Warning: Mitchell theorem does not apply to properties involving infinitely many objects.

Exercise 4.6. In $R-\mathbf{Mod}$, an infinite product of exact triples is exact, but in $\mathbf{Sh}\,(X)$ this is not true in general. We will see this later. Also $\mathbf{Qcoh}\,(X)$ for smooth projective variety X does not have any projective objects.

4.3 Projective and Injective Objects

Let $\mathcal{A}$ be an abelian category.

Definition 4.11. An object $P \in Ob\,(\mathcal{A})$ is projective if $\mathrm{Hom}_{\mathcal{A}}\,(P, -) : \mathcal{A} \to \mathbf{Ab}$ is exact.

Definition 4.12. An object $I \in Ob\,(\mathcal{A})$ is injective if $\mathrm{Hom}_{\mathcal{A}}\,(-, I) : \mathcal{A}^{op} \to \mathbf{Ab}$ is exact.

Remark 4.2. Projectives in $\mathcal{A}$ is equivalent to injectives in $\mathcal{A}^{op}$. But this does not mean that projectives and injectives have similar properties (since $\mathcal{A}$ and $\mathcal{A}^{op}$ are very different categories).

Note: $\mathrm{Hom}_{\mathcal{A}}\,(P, -)$ is always left exact for any $P \in Ob\,(\mathcal{A})$, so P is projective means that $\mathrm{Hom}_{\mathcal{A}}\,(P, -)$ is right exact.

Claim 4.1. The following are equivalent:

1. P is projective.

2. For any $X \to Y \to Z \to 0$ exact, $\mathrm{Hom}_{\mathcal{A}}\,(P, X) \to \mathrm{Hom}_{\mathcal{A}}\,(P, Y) \to \mathrm{Hom}_{\mathcal{A}}\,(P, Z) \to 0$ is exact.

3. For any $Y \to Z \to 0$ epi, $\mathrm{Hom}_{\mathcal{A}}\,(P, Y) \to \mathrm{Hom}_{\mathcal{A}}\,(P, Z) \to 0$ is epi.

4. For any $Y \to Z \to 0$ epi and any $\varphi : P \to Z$, there exists $\tilde{\varphi}$ such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{\varphi}"'] \arrow[d, "\varphi"] & \\ Y \arrow[r, two heads] & Z \arrow[r] & 0 \end{tikzcd}$$

commutes.

We have similar result for injectives.

Claim 4.2. The following are equivalent:

1. I is injective.
2. For any $0 \to X \to Y$ exact, $\mathrm{Hom}_{\mathcal{A}}(Y, I) \to \mathrm{Hom}_{\mathcal{A}}(X, I) \to 0$ is exact.
3. For any $0 \to X \to Y$ mono, $\mathrm{Hom}_{\mathcal{A}}(Y, I) \to \mathrm{Hom}_{\mathcal{A}}(X, I) \to 0$ is epi.
4. For any $0 \to X \to Y$ mono and any $\varphi : X \to I$, there exists $\tilde{\varphi}$ such that

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook] \arrow[dr, "\varphi"'] & Y \arrow[d, dashed, "\tilde{\varphi}"'] \\ & & I \end{tikzcd} \tag{23}$$

commutes.

Lemma 4.2.

1. *$P \in Ob(\mathcal{A})$ is projective if and only if for any $p : X \twoheadrightarrow P$ there is an isomorphism $X \cong X' \oplus P$ such that p factors as*

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[rr, bend right, "p"'] & X' \oplus P \arrow[r, two heads, "can"] & P \end{tikzcd}$$

2. *$I \in Ob(\mathcal{A})$ is injective if and only if for any $i : I \hookrightarrow X$ there is an isomorphism $X \cong X' \oplus I$ such that i factors as*

$$\begin{tikzcd} I \arrow[r, hook, "can"] \arrow[rr, bend right, "i"'] & X' \oplus I \arrow[r, "\sim"] & X \end{tikzcd}$$

Proof. We will prove (1). The proof of (2) is similar.

If P is projective, given $p : X \twoheadrightarrow P$, consider

$$\begin{tikzcd} & P \arrow[dl, dashed, "i"'] \arrow[d, equal, "\mathrm{Id}"] & \\ X \arrow[r, two heads, "p"] & P \arrow[r] & 0 \end{tikzcd}$$

such that $pi = \mathrm{Id}_P$, so i is mono. Put $X' = \mathrm{Ker}\,(p)$ with $i' : X' \hookrightarrow X$ being the canonical inclusion. Note $pi' = 0$.

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, hook, "i'"] & X \arrow[r, shift left, "p"] & P \arrow[l, shift left, "i"] \arrow[r] & 0 \end{tikzcd}$$

Note $p\,(\mathrm{Id}_X - ip) = p - pip = p - p = 0$, $(\mathrm{Id}_X - ip) \in \mathrm{Ker}\,(p_*) = \mathrm{Im}\,(i'_*)$, so there exists $p' : X \to X'$ such that $\mathrm{Id}_X - ip = i'p'$.

We claim that $p'i' = \mathrm{Id}_{X'}$ and $p'i = 0$. $i'p'i' = (\mathrm{Id}_X - ip)\,i' = i' - ipi' = i'$, so $i'\,(p'i' - \mathrm{Id}_{X'}) = 0$, and since i' is mono, $p'i' = \mathrm{Id}_{X'}$. $i'p'i = (\mathrm{Id}_X - ip)\,i = i - ipi = 0$, and since i' is mono, $p'i = 0$. Thus we get a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, shift left, "i'"] & X \arrow[l, shift left, "p'"] \arrow[r, shift left, "p"] & P \arrow[l, shift left, "i"] \arrow[r] & 0 \end{tikzcd}$$

so $X = X' \oplus P$.

Assume $\pi : Y \to Z$ is onto and $\varphi : P \to Z$. Put $X \coloneqq \mathrm{Ker}\,[(-\pi, \varphi) : Y \oplus P \to Z]$ and define $\left[X \xrightarrow{p} P\right] \coloneqq [X \hookrightarrow Y \oplus P \twoheadrightarrow P]$ and $\left[X \xrightarrow{q} Y\right] \coloneqq [X \hookrightarrow Y \oplus P \twoheadrightarrow Y]$. Since π is onto, p is onto (indeed, if $x \in P$, then there exists $y \in Y$ such that $\pi\,(y) = \varphi\,(x)$, take $(y, x) \in X$ and $p\,(y, x) = x$). By assumption, p splits, so there exists $s : P \to X$ such that $ps = \mathrm{Id}_P$. Define $\tilde{\varphi} = qs$, then $\pi\tilde{\varphi} = \varphi$ as required. $\square$

Definition 4.13. An object $Y \in Ob\,(\mathcal{A})$ is an *extension* of $Z \in Ob\,(\mathcal{A})$ by $X \in Ob\,(\mathcal{A})$ if there exists an exact triple $0 \to X \to Y \to Z \to 0$.

Notation. $\mathbf{Proj}\,(\mathcal{A})$ is the class of projective objects in $\mathcal{A}$. $\mathbf{Inj}\,(\mathcal{A})$ is the class of injective objects in $\mathcal{A}$.

Exercise 4.7. Show that

1. $\mathbf{Proj}\,(\mathcal{A})$ and $\mathbf{Inj}\,(\mathcal{A})$ are closed under extension and direct sums.

2. $\mathbf{Proj}\,(\mathcal{A})$ and $\mathbf{Inj}\,(\mathcal{A})$ are closed under taking direct summands.

3. $\mathbf{Proj}\,(\mathcal{A})$ is closed under taking kernels of epimorphisms, i.e. if $p : P_1 \twoheadrightarrow P_2$ with P_1, P_2 projective, then $\mathrm{Ker}\,(p)$ is projective.

4. $\mathbf{Inj}(\mathcal{A})$ is closed under taking cokernels of monomorphisms, i.e. if $i : I_1 \hookrightarrow I_2$ with I_1, I_2 injective them $\mathrm{Coker}(i)$ is injective.

Proof. (1) and (2) are straightforward. We only need to prove (3), and then (4) is similar.

Consider the short exact sequence

$$0 \longrightarrow P_0 \xrightarrow{i} P_1 \xrightarrow{p} P_2 \longrightarrow 0$$

where $P_0 = \mathrm{Ker}(p)$, it splits because P_2 is projective, and we have

$$\begin{tikzcd} 0 \arrow[r] & P_0 \arrow[r, shift left, "i"] \arrow[d, dashed] \arrow[dr, "\varphi"] & P_1 \arrow[l, dashed, shift left, "s"] \arrow[r, shift left, "p"] & P_2 \arrow[l, dashed, shift left, "t"] \arrow[r] & 0 \\ & Y \arrow[r, two heads, "\pi"'] & Z \arrow[r] & 0 & \end{tikzcd}$$

For any $\varphi : P_0 \to Z$, $\varphi \circ s : P_1 \to Z$ extends to $\phi : P_1 \to Y$, then $\tilde{\varphi} = \phi \circ i : P_0 \to Y$ is the desired extension since

$$\pi \circ \tilde{\varphi} = \pi \circ \phi \circ i = \varphi \circ s \circ i = \varphi.$$

□

4.4 Projective and Injective Modules

Definition 4.14. An abelian category $\mathcal{A}$ has enough projectives if for any $X \in Ob(\mathcal{A})$, there exists $P \twoheadrightarrow X$ epi with P projective.

Definition 4.15. An abelian category $\mathcal{A}$ has enough injectives if for any $X \in Ob(\mathcal{A})$, there exists $X \hookrightarrow I$ mono with I injective.

Theorem 4.4. *Let R be a unital associative ring. $R-\mathbf{Mod}$ and $\mathbf{Mod}-R$ has enough projective and injective modules.*

Projective Modules

Lemma 4.3. *Every free module is projective.*

Proof. The forgetful functor $U : R - \mathbf{Mod} \to \mathbf{Set}$ has left adjoint, the free functor

$$\begin{array}{rccc} R\langle - \rangle : & \mathbf{Set} & \rightleftharpoons & R - \mathbf{Mod} \quad : U \\ & S & \mapsto & R\langle S \rangle \end{array}$$

where $R\langle S\rangle$ is the free R-module based on S. $\mathrm{Hom}_R(R\langle S\rangle, M) \cong \mathrm{Hom}_{\mathbf{Set}}(S, M)$.

Sets are free in the sense that for any surjective $p : X \twoheadrightarrow Y$, $p_* : \mathrm{Hom}(S, X) \to \mathrm{Hom}(S, Y)$ is surjective.

$$\begin{tikzcd} & S \arrow[dl, dashed, "\tilde{\varphi}"'] \arrow[d, "\varphi"] & \\ X \arrow[r, two heads, "p"] & Y \arrow[r] & 0 \end{tikzcd}$$

If M, N are R-modules, $p : M \to N$ is epi, then $Up : UM \twoheadrightarrow UN$ is onto, so

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Set}}(S, UM) \arrow[r, two heads] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbf{Set}}(S, UN) \arrow[d, "\cong"] \\ \mathrm{Hom}_R(R\langle S\rangle, M) \arrow[r, two heads] & \mathrm{Hom}_R(R\langle S\rangle, N) \end{tikzcd}$$

Hence $\mathrm{Hom}_R(R\langle S\rangle, -)$ is exact, so $R\langle S\rangle$ is projective. □

Corollary 4.2. *P is a projective R-module if and only if P is a direct summand of a free module, i.e. there exists $Q \in Ob(R - \mathbf{Mod})$ such that $P \oplus Q = R\langle S\rangle$.*

Corollary 4.3. *$R - \mathbf{Mod}$ has enough projectives.*

Proof. For any $M \in Ob(R - \mathbf{Mod})$, $R\langle UM\rangle \twoheadrightarrow M$. □

Remark 4.3. The problem of describing projectives for a specific R is not trivial.

Exercise 4.8. If X is an affine variety over a field k, then for $R = \mathcal{O}(X)$ the coordinate ring, $\mathcal{O}(X) - \mathbf{Mod} \cong \mathbf{Qcoh}(X)$ the category of quasi-coherent sheaves on X.

$$\begin{tikzcd} \mathcal{O}(X) - \mathbf{Mod} & \mathbf{Qcoh}(X) \arrow[l, "\cong"'] \\ \mathcal{O}(X) - \mathbf{Mod}^{\mathrm{fg}}(X) \arrow[u, hook] & \mathbf{coh}(X) \arrow[l, "\cong"'] \arrow[u, hook] \\ \mathbf{Proj}\left(\mathcal{O}(X) - \mathbf{Mod}^{\mathrm{fg}}\right)(X) \arrow[u, hook] & \mathbf{VB}(X) \arrow[l, "\cong"'] \arrow[u, hook] \end{tikzcd}$$

where $\mathbf{VB}(X) \cong \{\text{locally free finite rank sheaves on } X\}$.

Question 1: How to distinguish nontrivial (non free) projective modules from trivial ones.

Let $\mathbb{P}(R)$ be the subcategory of finitely generated projective modules in $R-\mathbf{Mod}$. The Grothendieck group of R is defined by

$$K_0(R) := \mathbb{Z}\langle \text{iso-classes of f.g. projectives}\rangle / \langle [P] = [P_1] + [P_2] \,|\, 0 \to P_1 \to P \to P_2 \to 0 \text{ exact}\rangle$$

P_1 is stably isomorphic to P_2 if $[P_1] = [P_2]$ in $K_0(R)$, i.e. there exists $Q \in Ob(\mathbb{P}(R))$ such that $P_1 \oplus Q \cong P_2 \oplus Q$. Note we can always choose Q to be free.

Definition 4.16. P is called stably free if P is stably isomorphic to a free module, i.e. $P \oplus R^n \cong R^m$.

Remark 4.4. Stably free need not to be free.

Example 4.10. Let $R = \mathcal{O}(S^2)$ be the coordinate ring of real 2-sphere S^2. $R = \mathbb{R}[x,y,z] / (x^2+y^2+z^2-1)$. Define a R-module homomorphism

$$\begin{array}{rccc} \pi: & R^3 & \twoheadrightarrow & R \\ & (\alpha,\beta,\gamma) & \mapsto & \alpha x + \beta y + \gamma z \end{array}$$

Notice that $\pi(x,y,z) = 1$ in R. This means that π has a section

$$\begin{array}{rccc} s: & R & \hookrightarrow & R^3 \\ & r & \mapsto & (rx, ry, rz) \end{array}$$

and $\pi \circ s = \mathrm{Id}_R$. Put $P = \mathrm{Ker}(\pi)$, then $P \oplus R \cong R^3$, so P is stably free of rank 2.

Claim. P is not free.

Proof. (Topological). Based on Hairy Ball theorem.

Assume that P is free, then $P \cong R^2$ and $R^2 \oplus R \cong R^3$, then we can choose a new basis of R^3 $\{e_1 = (x,y,z), e_2 = (f,g,h), e_3 = (f',g',h')\}$ where $f, f', g, g', h, h' \in R$ can be considered as polynomial functions on S^2. Then

$$d = \det \begin{vmatrix} x & f & f' \\ y & g & g' \\ z & h & h' \end{vmatrix} \in R^{\times}$$

is a unit.

Define a continuous vector field on S^2: $v \mapsto (f(v), g(v), h(v)) \in R^3$. Since $d \neq 0$ on S^2, this vector field is nowhere vanishing. If we project orthogonally this vector field to each tangent plane to S^2, we can get non-vanishing vector field tangent to S^2 which contradicts with Hairy Ball theorem. $\square$

Exercise 4.9. Show that for any commutative R, a finitely generated stably free module of rank 1 must be free. $P \oplus R^n \cong R^{n+1}$ for some n.

Hint: use Λ_R^n exterior power functor.

Question 2: Given a ring R, how to describe (find and classify up to honest isomorphism) finitely generated projective modules over R?

Serre conjecture: all projectives over R are free. Equivalently, there are no nontrivial algebraic vector bundles on $\mathbb{A}_k^n$.

Quillen and Suslin proved this in 1976. Proof is long and elementary.

Example 4.11. (Weyl algebra) Let k be a field with $\operatorname{char}(k) = 0$. Extend $k[x_1, \cdots, x_n]$ by adding derivatives.

$$A_n(k) = k\langle x_1, \cdots, x_n, \partial_1, \cdots, \partial_n\rangle / ([x_i, x_j] = [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{ij})_{1 \leq i,j \leq n}$$

When $n = 1$, $A_1(k) = \frac{k\langle x, \partial\rangle}{(\partial x - x\partial = 1)}$.

$K_0(A_n(k)) = \mathbb{Z}$, hence every projective is stably free.

Theorem 4.5. *[Stafford, 1981] Every stably free $A_n(k)$-module of rank ≥ 2 is free, but for each $n \geq 1$, there is no stably free module of rank 1.*

For $A_1(k)$, there are classified by Wilson, B in 1999. For $A_n(k), n \geq 2$, this is open.

Problem: Given a left ideal $I \subseteq A_n(k)$, find conditions when I is projective.

Exercise 4.10. Let $R = A_1(k)$, show that $I = Rx^{m+1} + R(x\partial + m)$ is projective but not free, $m \geq 1$.

Injective Modules

Proposition 4.1. *(Baer Criterion) I is injective in $R-$**Mod** if and only if 23 holds for left ideals in R, i.e. for any left ideal $J \subseteq R$ and any $\varphi : J \to I$ there exists $\tilde{\varphi} : R \to I$ such that $\tilde{\varphi}_J = \varphi$.*

Proof. Take $X \subseteq Y$ and $\varphi : X \to I$, we need to extend φ to $\tilde{\varphi} : Y \to I$.

Consider the set $\mathscr{S} = \{(X', \varphi' : X' \to I) \, | X \subseteq X' \subseteq Y, \varphi'|_X = \varphi\}$. Note that $\mathscr{S}$ is a poset with respect to inclusion and every chain is bounded above by Y. By Zorn's lemma, there exists (X_0, φ_0) a maximal element in $\mathscr{S}$. We claim that $X_0 = Y$. Assume that $X_0 \subsetneq Y$, take $y \in Y \backslash X_0$, define $X_1 = X_0 + Ry \subseteq Y$. It comes with projection

$$\begin{array}{rccc} p: & X_0 \oplus R & \to & X_1 \\ & (x_0, 0) & \mapsto & x_0 \\ & (0, r) & \mapsto & ry \end{array}$$

Take $J = \operatorname{Ker}(p)$, then we have a exact sequence in $R-$**Mod**

$$0 \to J \xrightarrow{(i_1, i_2)} X_0 \oplus R \xrightarrow{p} X_1 \to 0 \tag{24}$$

Note that $p|_{X_0} : X_0 \hookrightarrow X_1$ is mono, so $i_2 : J \to R$ is injective. (Indeed, if $a, b \in J, a \neq b$, then $(i_1(a), i_2(a)) \neq (i_1(b), i_2(b))$. If $i_1(a) = i_1(b)$, then $i_2(a) \neq i_2(b)$. . If $i_1(a) \neq i_1(b)$, since $p(i_1(a), i_2(a)) = p(i_1(b), i_2(b)) = 0$, $p(0, i_2(b)) = -p(i_1(b)) \neq -p(i_1(a), 0) = p(0, i_2(a))$, so $i_2(a) \neq i_2(b)$. Hence $i_2 : J \to R$ is injective.)

We can identify J as a left ideal in R via i_2. Since 24 is right exact, we have $X_1 \cong X_0 \coprod_J R$, and a Cocartesian diagram

$$\begin{tikzcd}
J \arrow[r, hook, "i_2"] \arrow[d] & R \arrow[d] \arrow[ddr, dashed, bend left, "\tilde{\psi}"] & \\
X_0 \arrow[r, hook] \arrow[drr, bend right, "\varphi_0"'] & X_1 \arrow[dr, dashed, "\varphi_1"] & \\
 & & I
\end{tikzcd}$$

Consider $\psi : J \xrightarrow{i_1} X_0 \xrightarrow{\varphi_0} I$, and by assumption we can extend it to $\tilde{\psi} : R \to I$ such that $\tilde{\psi}|_J = \psi$. Since $\tilde{\psi}$ and φ_0 agree on J, we have a canonical extension $\varphi_1 : X_1 = X_0 \coprod\limits_J R \to I$ such that $\varphi_1|_{X_0} = \varphi_0$.

The existence of φ_1 contradicts with the maximality of (X_0, φ_0), so $X_0 = Y$, and $\varphi_0 : Y \to I$ is the desired extension. $\square$

Next we will discuss properties of injectives from another aspect.

Definition 4.17. A nonzero element $0 \neq r \in R$ is called (left) regular if $R_r : R \to R, x \mapsto xr$ is injective, i.e. $xr = 0$ implies $x = 0$.

Definition 4.18. A left module M is called divisible if $rM = M$ for any left regular $r \in R$, i.e. for any $m \in M$ and any regular $r \in R$, there exists $m' \in M$ such that $rm' = m$.

Note that a quotient of a divisible module is divisible.

Example 4.12. $R = \mathbb{Z}$, $M = \mathbb{Q}$ is divisible as $\mathbb{Z}$-module, so is $\mathbb{Q}/\mathbb{Z}$.

Lemma 4.4. *For any R, injective R-modules are divisible.*

Proof. Take any regular $r \in R$, consider the exact sequence

$$0 \to R \xrightarrow{\cdot r} R \to R/R \cdot r \to 0$$

If I is injective, apply $\operatorname{Hom}_R(-, I)$ to this short exact sequence we get a new short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(R/R \cdot r, I) \arrow[r] & \operatorname{Hom}_R(R, I) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Hom}_R(R, I) \arrow[r] \arrow[d, "\cong"] & 0 \\
 & & I \arrow[r, "r\cdot"] & I &
\end{tikzcd}$$

hence $rI = I$. □

Lemma 4.5. *If R is a (left) PID, every divisible is injective.*

Proof. Take any left ideal $J \subseteq R$, since R is a (left) PID, there exists $r \in R$ such that $J = R \cdot r$. r is regular since R is a domain. Any $\varphi : J \to M$ is determined by $m = \varphi(r) \in M$, since $\varphi(x \cdot r) = x\varphi(r) = x \cdot m, \forall x \in R$. Take $m' \in M$ such that $r \cdot m' = m$, define

$$\begin{array}{rccc} \tilde{\varphi}: & R & \to & M \\ & x & \mapsto & xm' \end{array}$$

then $\tilde{\varphi}|_J = \varphi$ because $\tilde{\varphi}(xr) = x\tilde{\varphi}(r) = xrm' = xm$. □

Corollary 4.4. **Ab** *has enough injectives.*

Proof. $\mathbf{Ab} = \mathbb{Z} - \mathbf{Mod}$ where $\mathbb{Z}$ is PID. Note products and quotients of injectives are injectives. For any $\mathbb{Z}$-module M, define $I(M) = \prod_{\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$. $I(M)$ is injective, being a product of injectives, and there is a canonical map $i : M \hookrightarrow I(M)$. □

Proposition 4.2. *Let $F : \mathcal{A} \rightleftharpoons \mathcal{B} : G$ be a pair of additive adjoint functors between abelian categories, then*

1. *If F is (left) exact, then G maps injectives to injectives.*
2. *If G is (right) exact, then F maps projectives to projectives.*

Proof. For any $I \in Ob(\mathcal{B})$, consider $\mathrm{Hom}(-, GI) : \mathcal{A}^{op} \to \mathbf{Ab}$,

$$\mathrm{Hom}_{\mathcal{A}}(-, GI) \cong \mathrm{Hom}_{\mathcal{B}}(F(-), I) = \mathrm{Hom}_{\mathcal{B}}(-, I) \circ F$$

If I is injective and F is exact, then $\mathrm{Hom}_{\mathcal{B}}(-, I) \circ F$ is exact, so $G(I)$ is injective. □

Recall associated to the canonical homomorphism of rings $i : \mathbb{Z} \to R$ is the adjoint triple

$(i^*, i_*, i^!)$)

$$\begin{tikzcd} R-\mathbf{Mod} \arrow[d, "i_*"] \\ \mathbb{Z}-\mathbf{Mod} \arrow[u, bend left=60, "i^*"] \arrow[u, bend right=60, "i^!"'] \end{tikzcd}$$

where i_* is the restriction functor given by $\mathbb{Z} \to R \to M$, and

$$\begin{aligned} i^* : \quad \mathbb{Z}-\mathbf{Mod} &\to R-\mathbf{Mod} \\ M &\mapsto R \otimes_{\mathbb{Z}} M \end{aligned}$$

is the induction functor, and

$$\begin{aligned} i^! : \quad \mathbb{Z}-\mathbf{Mod} &\to R-\mathbf{Mod} \\ M &\mapsto \mathrm{Hom}_{\mathbb{Z}}(R, M) \end{aligned}$$

is the coinduction functor. Then i^* is right exact, i_* is exact and $i^!$ is left exact. By the above proposition, $i^!$ maps injectives to injectives. Also $i^!$ maps monics to monics, and the unit adjunction associated to $(i_*, i^!)$ is monic since for any R-module M,

$$\eta_M : M \hookrightarrow i^! i_* M$$

is a monomorphism by considering

$$\eta_M : M \cong \mathrm{Hom}_R(R, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(R, M) \cong i^! i_*(M)$$

Finally, we can combine all the these to prove the following theorem.

Theorem 4.6. $R-\mathbf{Mod}$ *has enough injectives.*

Proof. For any R-module M, $i_*(M)$ is a $\mathbb{Z}$-module. Since **Ab** has enough injectives, there exists an injective $\mathbb{Z}$-module I such that $f : i_* M \hookrightarrow I$ is a monomorphism in **Ab**. Apply $i^!$ to f, we get

$$M \hookrightarrow i^! i_*(M) \hookrightarrow i^!(I)$$

and $i^!(I)$ is injective in $R-\mathbf{Mod}$. Hence M embeds in an injective in $R-\mathbf{Mod}$. $\square$

4.5 Injective Envelopes and Projective Covers

Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives.

Definition 4.19. An injective envelope (hull) of $M \in Ob(\mathcal{A})$ is a monomorphism $i : M \hookrightarrow E$ in $\mathcal{A}$ with

1. $E \in \mathbf{Inj}(\mathcal{A})$.
2. For any monomorphism $\varphi : M \to I$ with I injective, there exists $\bar{\varphi} : E \hookrightarrow I$ such that

$$\begin{tikzcd} M \arrow[rr, hook, "\varphi"] \arrow[dr, hook, "i"'] & & I \\ & E \arrow[ur, hook, "\bar{\varphi}"'] & \end{tikzcd}$$

commutes.

We say that $\mathcal{A}$ has injective envelopes if (E, i) exists for every $M \in Ob(\mathcal{A})$.

Remark 4.5. $\bar{\varphi}$ needs not to be unique. If (E, i) exists, it is unique up to non-unique isomorphism.

Dually , assume $\mathcal{A}$ has enough projectives.

Definition 4.20. A *projective cover* of $M \in Ob(\mathcal{A})$ is an epimorphism $p : P \twoheadrightarrow M$ in $\mathcal{A}$ with

1. $P \in \mathbf{Proj}(\mathcal{A})$.
2. For any epimorphism $\psi : Q \twoheadrightarrow M$ with Q projective, there exists $\bar{\psi} : Q \twoheadrightarrow P$ such that

$$\begin{tikzcd} Q \arrow[rr, two heads, "\psi"] \arrow[dr, two heads, "\bar{\psi}"'] & & M \\ & P \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

commutes.

We say that $\mathcal{A}$ has projective covers if (P, p) exists for every $M \in Ob(\mathcal{A})$.

There is a useful characterization of injective envelopes in terms of essential extensions.

Definition 4.21. A monomorphism $f : L \hookrightarrow E$ is called essential (or E is called an essential extension of L) if for any nonzero subobject $0 \neq M \subseteq E$, $f(L) \cap M \neq \emptyset$.

Lemma 4.6. *If $f : L \hookrightarrow E$ and E is injective, then E is an essential extension of L if and only if (E, f) is an injective envelope.*

Proof. If (E, f) is an injective envelope, for any subobject $0 \neq M \subseteq E$, we have a canonical monomorphism $i : M \hookrightarrow E$. If $I = f(L) \cap M = \emptyset$, Let $C = \mathrm{Coker}(i)$, then the composition

$$L \xhookrightarrow{f} E \xrightarrow{c} C$$

is mono since if $c \circ f \circ g = c \circ f \circ h$ for $g, h : A \to L$,

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & L \arrow[r, hook, "f"] & E \arrow[r, two heads, "c"] & C \end{tikzcd}$$

$c \circ f \circ (g - h) = 0$, note $f((g-h)(A)) \cap M = \emptyset$, so $f \circ (g - h) = 0$, and since f is mono, $g - h = 0$, i.e. $g = h$. This contradicts with the fact that (E, f) is an injective envelope, so $f(L) \cap M \neq \emptyset$.

If E is an essential extension of L, for any a monomorphism $\varphi : L \hookrightarrow I$ with I injective, we have

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook, "f"] \arrow[d, hook, "\varphi"] & E \arrow[dl, "\bar{\varphi}"] \\ & I & \end{tikzcd}$$

Suppose $\bar{\varphi}$ is not mono, then $M = \mathrm{Ker}(\bar{\varphi}) \neq 0$, and $f(L) \cap M = \emptyset$, which contradicts with the fact that E is an essential extension of L, so $\bar{\varphi}$ is mono, hence (E, f) is an injective envelope. □

Thus injective envelopes are maximal essential extensions.

Corollary 4.5. *If $\mathcal{A}$ has injective envelopes, then $E \in$ **Inj** $(\mathcal{A})$ if and only if any essential extension $i : E \to E'$ is an isomorphism.*

Exercise 4.11. Show that $k = k[x]/(x)$ has no projective covers in $k[x] -$ **Mod**.

Proof. Consider $Q = M_n(k)$ which is simple, there is no epimorphism $Q \twoheadrightarrow k$. □

Corollary 4.6. $R -$ **Mod** *has injective envelopes buy not projective covers.*

Remark 4.6. Even if $\mathcal{A}$ has enough injectives, injective envelopes may not exist.

Example 4.13. $\mathcal{A} = (k[x] - \mathbf{Mod})^{op}$ has no injective envelopes.

4.6 Main Theorem of Homological Algebra

Let $\mathcal{A}$ be an abelian category, $\mathbf{Com}(\mathcal{A})$ is the category of complexes over $\mathcal{A}$.

Recall that $\mathbf{Com}(\mathcal{A})$ is abelian with kernel and cokernel defined fibre-wise. If $f^\bullet : X^\bullet \to Y^\bullet$ is a morphism in $\mathbf{Com}(\mathcal{A})$, then

$$\operatorname{Ker}(f^\bullet) \coloneqq \left[\cdots \to \operatorname{Ker}(f^n) \xrightarrow{d^n|_{\ker(f^n)}} \operatorname{Ker}\left(f^{n+1}\right) \xrightarrow{d^{n+1}|_{\ker\left(f^{n+1}\right)}} \operatorname{Ker}\left(f^{n+2}\right) \to \cdots \right]$$

Let $A^\bullet, B^\bullet, C^\bullet$ be three complexes in $\mathbf{Com}(\mathcal{A})$, then by definition, $0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$ is exact in $\mathbf{Com}(\mathcal{A})$ if and only if $0 \to A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \to 0$ is exact in $\mathcal{A}$ for any $n \in \mathbb{Z}$.

Define $\mathbf{Exc}(\mathcal{A})$ to be the category with

Objects: short exact complexes $\left\{0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0\right\}$ in $\mathbf{Com}(\mathcal{A})$.

Morphisms: commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & A_1^\bullet \arrow[r, "f_1^\bullet"] \arrow[d, "\varphi_A^\bullet"] & B_1^\bullet \arrow[r, "g_1^\bullet"] \arrow[d, "\varphi_B^\bullet"] & C_1^\bullet \arrow[r] \arrow[d, "\varphi_C^\bullet"] & 0 \\
0 \arrow[r] & A_2^\bullet \arrow[r, "f_2^\bullet"] & B_2^\bullet \arrow[r, "g_2^\bullet"] & C_2^\bullet \arrow[r] & 0
\end{tikzcd}$$

in $\mathbf{Com}(\mathcal{A})$.

Recall we defined the n-th cohomology functor to be

$$\begin{aligned}
H^n : \quad \mathbf{Com}(\mathcal{A}) &\longrightarrow \mathcal{A} \\
(X^\bullet, d^\bullet) &\longmapsto H^n(X) \coloneqq \operatorname{Ker}(d^n)/\operatorname{Im}\left(d^{n-1}\right) \cong \operatorname{Coker}\left(\operatorname{Im} d^{n-1} \to \operatorname{Ker} d^n\right) = \operatorname{Ker}\left(\operatorname{Coker} d^{n-1} \to \operatorname{Coim} d^n\right)
\end{aligned}$$

We can assemble H^n into one additive functor

$$\begin{aligned}
H^\bullet : \quad \mathbf{Com}(\mathcal{A}) &\longrightarrow \mathbf{Com}(\mathcal{A}) \\
(X^\bullet, d^\bullet) &\longmapsto (H^n(X), 0)_{n \geq 0}
\end{aligned}$$

Define for each $n \in \mathbb{Z}$ two functors $F^n, G^n : \mathbf{Exc}\,(\mathcal{A}) \to \mathcal{A}$ given by

$$\begin{aligned} F^n\left(0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0\right) &:= H^n\,(C) \qquad F^n\left(\varphi_A^\bullet, \varphi_B^\bullet, \varphi_C^\bullet\right) := H^n\,(\varphi_C) \\ G^n\left(0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0\right) &:= H^{n+1}\,(A) \quad G^n\left(\varphi_A^\bullet, \varphi_B^\bullet, \varphi_C^\bullet\right) := H^{n+1}\,(\varphi_A) \end{aligned}$$

Lemma 4.7. *There is a natural transformation $\delta^n : F^n \to G^n$ for each $n \in \mathbb{Z}$, called the* connecting homomorphism.

Proof. Given $E = (0 \to A \to B \to C \to 0) \in Ob\,(\mathbf{Exc}\,(A))$, we need to define $\delta^n\,(E) : H^n\,(C) \to H^{n+1}\,(A)$. Apply Mitchell Theorem and think of objects A, B, C as complexes of modules over a ring.

Let's construct $\delta^n\,(E)$ explicitly.

$$\begin{tikzcd}
0 \arrow[r] & A^{n-1} \arrow[r, "f^{n-1}"] \arrow[d, "d_A^{n-1}"] & B^{n-1} \arrow[r, "g^{n-1}"] \arrow[d, "d_B^{n-1}"] & C^{n-1} \arrow[r] \arrow[d, "d_C^{n-1}"] & 0 \\
0 \arrow[r] & A^{n} \arrow[r, "f^{n}"] \arrow[d, "d_A^{n}"] & B^{n} \arrow[r, "g^{n}"] \arrow[d, "d_B^{n}"] & C^{n} \arrow[r] \arrow[d, "d_C^{n}"] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r, "f^{n+1}"] \arrow[d, "d_A^{n+1}"] & B^{n+1} \arrow[r, "g^{n+1}"] \arrow[d, "d_B^{n+1}"] & C^{n+1} \arrow[r] \arrow[d, "d_C^{n+1}"] & 0 \\
0 \arrow[r] & A^{n+2} \arrow[r, "f^{n+2}"] & B^{n+2} \arrow[r, "g^{n+2}"] & C^{n+2} \arrow[r] & 0
\end{tikzcd}$$

Given $z \in H^n\,(C)$, choose $c \in C^n$ such that $d_C^n\,(c) = 0$ and $z = [c]$. Since g^n is epic, there exists $b \in B^n$ such that $g^n\,(b) = c$. Note that $g^{n+1}\,(d_B^n\,(b)) = d_C^n\,(g^n\,(b)) = 0$, so $d_B^n\,(b) \in \mathrm{Ker} g^{n+1} = \mathrm{Im} f^{n+1}$, there exists a unique $a \in A^{n+1}$ such that $f^{n+1}\,(a) = d_B^n\,(b)$. Note that $f^{n+2}\left(d_A^{n+1}\,(a)\right) = d_B^{n+1}\left(f^{n+1}\,(a)\right) = d_B^{n+1} d_B^n\,(b) = 0$, and since f^{n+2} is mono, $d_A^{n+1}\,(a) = 0$, so a is a $(n+1)$-cycle. Define $\delta^n\,(E)\,(z) = [a]$.

This doesn't depend on the choice of c. Suppose $z = [c']$, then $c' = c + d_C^{n-1}\,(c'')$ for some $c'' \in C^{n-1}$. Then there exists $b'' \in B^{n-1}$ such that $g^{n-1}\,(b'') = c''$. By commutativity, $g^n\left(d_B^{n-1}\,(b'')\right) = d_C^{n-1}\,(c'')$, and $d_B^n\left(d_B^{n-1}\,(b'')\right) = 0$, since f^{n+1} is injective, $f^{n+1}\,(0) = 0$, so $\delta^n\,(E)\,([c']) = [a]$. ☐

Theorem 4.7. *For any exact sequence*

$$E = [0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0] \ni \mathbf{Exc}\,(A)$$

the sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{A}) \arrow[r] & H^0(\mathcal{B}) \arrow[r] & H^0(\mathcal{C}) \arrow[dll, "\alpha^0"] & \\
 & H^1(\mathcal{A}) \arrow[r] & H^1(\mathcal{B}) \arrow[r] & H^1(\mathcal{C}) \arrow[dll, "\alpha^1"] & \\
 & H^2(\mathcal{A}) \arrow[r] & H^2(\mathcal{B}) \arrow[r] & H^2(\mathcal{C}) \arrow[dll, dashed] & \\
 & H^n(\mathcal{A}) \arrow[r] & H^n(\mathcal{B}) \arrow[r] & H^n(\mathcal{C}) \arrow[r] & \cdots
\end{tikzcd}$$

is acyclic, or exact. It's called long exact cohomology sequence associated to E.

Proof. Routine verification. □

Remark 4.7. The long exact sequence is functorial in the sense that $E \to LE$ is a functor $\mathbf{Exc}\,(\mathcal{A}) \to \mathbf{Com}\,(\mathcal{A})$.

If we invert cohomological algebra to homological algebra $\tilde{X}_n = X^{-n}$, then $0 \to \tilde{\mathcal{A}} \to \tilde{\mathcal{B}} \to \tilde{\mathcal{C}} \to 0$ gives

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n+1}(\mathcal{A}) \arrow[r] & H^{n+1}(\mathcal{B}) \arrow[r] & H^{n+1}(\mathcal{C}) \arrow[dll, "\alpha^0"] & \\
 & H^n(\mathcal{A}) \arrow[r] & H^n(\mathcal{B}) \arrow[r] & H^n(\mathcal{C}) \arrow[dll, "\alpha^1"] & \\
 & H^{n-1}(\mathcal{A}) \arrow[r] & H^{n-1}(\mathcal{B}) \arrow[r] & H^{n-1}(\mathcal{C}) \arrow[dll, dashed] & \\
 & H^0(\mathcal{A}) \arrow[r] & H^0(\mathcal{B}) \arrow[r] & H^0(\mathcal{C}) \arrow[r] & 0
\end{tikzcd}$$

Lemma 4.8. *(Snake Lemma) Given commutative diagram in $\mathcal{A}$*

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\alpha"] & B_1 \arrow[r] \arrow[d, "\beta"] & C_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0
\end{tikzcd}$$

with exact rows, then we have

$$\begin{tikzcd}
\cdots \arrow[r] & Ker(\alpha)(\mathcal{A}) \arrow[r] & Ker(\beta) \arrow[r] & H^{n+1}(\mathcal{C}) \arrow[dll, "\alpha^0"] & \\
 & H^n(\mathcal{A}) \arrow[r] & H^n(\mathcal{B}) \arrow[r] & H^n(\mathcal{C}) \arrow[r] & 0
\end{tikzcd}$$

Corollary 4.7. *(Five Lemma) If we have*

$$
\begin{tikzcd}
X_1 \arrow[r] \arrow[d, two heads, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2", "\wr"'] & X_3 \arrow[r] \arrow[d, "f_3"] & X_4 \arrow[r] \arrow[d, "f_4", "\wr"'] & X_1 \arrow[d, hook, "f_1"] \\
Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & Y_4 \arrow[r] & Y_5
\end{tikzcd}
$$

then f_3 is an isomorphism if the two rows are exact, and f_1 is epi, f_5 is mono, f_2, f_4 are isomorphisms.

Remark 4.8. If $X_1 = Y_1 = X_5 = Y_5 = 0$, this immediately follows from snake lemma.

Corollary 4.8. *(3×3 lemma) In the following diagram,*

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_2 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_3 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

If the rows and middle column are exact, then the left column is exact if and only if the right column is exact.

4.7 Operations on Complexes

There are various operations and constructions on complexes similar to one's on spaces.

Quasi-isomorphism and Homotopy Equivalences

Recall that a morphism $f : X^\bullet \to Y^\bullet$ is a quasi-isomorphism if $H^n(f) : H^n(X) \to H^n(Y)$ is an isomorphism in $\mathcal{A}$ for any $n \in \mathbb{Z}$.

Example 4.14. $X^\bullet$ is acyclic if and only if $0 : X^\bullet \to X^\bullet$ is a quasi-isomorphism.

Example 4.15. Given a complex $\left[0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0\right]$ in $\mathbf{Com}(\mathcal{A})$, define

$$\begin{array}{ccccccccc} X^\bullet & = & [0 & \to & A & \xrightarrow{f} & B & \to & 0] \\ & & & & \downarrow & & \downarrow g & & \\ Y^\bullet & = & [0 & \to & 0 & \to & C & \to & 0] \end{array}$$

then $\left[0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0\right]$ is exact if and only if $g^\bullet$ is a quasi-isomorphism.

Definition 4.22. $f, g : X^\bullet \to Y^\bullet$ are called *homotopic* in $\mathbf{Com}(\mathcal{A})$ if there exists $h^n : X^n \to Y^{n-1}$ such that $f^n - g^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$

$$\begin{tikzcd} \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, "g^{n-1}"] & X^n \arrow[r] \arrow[dl, "h^n"'] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] & X^{n+1} \arrow[r] \arrow[dl, "h^{n+1}"'] \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] & \cdots \\ \cdots \arrow[r] & Y^{n-1} \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

Definition 4.23. $f^\bullet : X^\bullet \to Y^\bullet$ is called *homotopy equivalence* in $\mathbf{Com}(\mathcal{A})$ if there exists $g^\bullet : Y^\bullet \to X^\bullet$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$.

Lemma 4.9. *If $f^\bullet \simeq g^\bullet$ then $H^\bullet(f) = H^\bullet(g)$.*

Proof. It suffices to prove that $f \simeq 0$ if and only if $H^\bullet(f) = 0$. $f \simeq 0$ implies that $f^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$ then $f^n|_{\mathrm{Ker} d_X^n} = d_Y^{n-1} h^n$ factors through $\mathrm{Im} d_Y^{n-1}$, so $H^n(f) = 0, \forall n \in \mathbb{Z}$. □

Corollary 4.9. *Every homotopy equivalence is quasi-isomorphism.*

Proof. $f^\bullet \circ g^\bullet \simeq \mathrm{Id}_Y$ if and only if $H^\bullet(f) \circ H^\bullet(g) = \mathrm{Id}$, and $g^\bullet \circ f^\bullet \simeq \mathrm{Id}_Y$ if and only if $H^\bullet(g) \circ H^\bullet(f) = \mathrm{Id}$. Hence $H^\bullet(f)$ and $H^\bullet(g)$ are quasi-isomorphisms. □

Remark 4.9. Not every quasi-isomorphism is a homotopy equivalence, for example, in $\mathcal{A} = \mathbf{Ab}$,

$$\begin{tikzcd} X^\bullet & = & {[0} \arrow[r] & \mathbb{Z} \arrow[r, "2\cdot"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & {0]} \\ Y^\bullet & = & {[0} \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & {0]} \end{tikzcd}$$

is a quasi-isomorphism but not a homotopy equivalence, since $\mathrm{Hom}(Y^\bullet, X^\bullet) = 0$.

This has important implication.

Remark 4.10. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Extend it degree-wise to

$$\begin{array}{rccc} F^\bullet : & \mathbf{Com}(\mathcal{A}) & \to & \mathbf{Com}(\mathcal{B}) \\ & (X^\bullet, d_X) & \mapsto & (FX^\bullet, Fd_X) \end{array}$$

Note that even when $F^\bullet$ is not exact, it maps homotopy equivalences to homotopy equivalences. But F does not preserve quasi-isomorphism.

Example 4.16. $\mathcal{A} = \mathbf{Ab}$, we modify the above example

$$\begin{tikzcd} X^\bullet \arrow[d, "f^\bullet"] & = & {[\cdots} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, hook] \arrow[d] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "2\cdot"] \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[r, hook] \arrow[d] & \mathbb{Z}/4\mathbb{Z} \arrow[r] \arrow[d] & 0] \\ Y^\bullet & = & {[\cdots} \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0] \end{tikzcd}$$

$f^\bullet$ is a quasi-isomorphism. Take $F = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -) : \mathbf{Ab} \to \mathbf{Ab}$ and apply F to $f^\bullet$ we get

$$\begin{tikzcd} FX^\bullet \arrow[d, "Ff^\bullet"] & = & {[\cdots} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "0"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "2\cdot"] \arrow[d, "0"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "0"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "0"] & 0] \\ FY^\bullet & = & {[\cdots} \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0] \end{tikzcd}$$

which is not a quasi-isomorphism.

4.8 Canonical Constructions on Complexes

Suspension

For each $k \in \mathbb{Z}$, define a functor $[k] : \mathbf{Com}(\mathcal{A}) \to \mathbf{Com}(\mathcal{A})$ with

Objects: $\left(X^\bullet[k], d^\bullet_{X[k]}\right)$ where $X[k]^n = X^{+k}$ and $d^n_{X[k]} = (-1)^k d^{n+k}_X$.

Morphisms: $[k]$ acts as identity.

Lemma 4.10. *$[k]$ is an automorphism of $\mathbf{Com}(\mathcal{A})$ with inverse $[-k]$.*

Example 4.17. There exists a natural embedding of categories

$$\begin{array}{rcl} \mathcal{A} & \longrightarrow & \mathbf{Com}\left(\mathcal{A}\right) \\ X & \longmapsto & \left[0 \to \underset{0}{X} \to 0\right] \end{array}$$

Note that $X\left[k\right] = \left[0 \to \underset{-k}{X} \to 0\right]$, $\left[k\right]$ shifts degree to the left $X\left[k\right]^n = \begin{cases} X & n = -k \\ 0 & n \neq -k \end{cases}$.

Remark 4.11. In homological notation, $X_\bullet \mapsto X\left[k\right]_\bullet$ is defined by $X\left[k\right]_n = X_{n-k}$. In this case,

$$X = \left[0 \to \underset{0}{X} \to 0\right] \longmapsto X\left[k\right] = \left[0 \to \underset{k}{X} \to 0\right] .$$

Mapping Cones

Let $\mathbf{Mor}\left(\mathbf{Com}\left(\mathcal{A}\right)\right)$ be the category of morphisms of complexes.

Objects: $\{f^\bullet : X^\bullet \to Y^\bullet\}$

Morphisms: $\mathrm{Hom}\left(f_1^\bullet, f_2^\bullet\right) = \left\{ \left(\alpha^\bullet, \beta^\bullet\right) \mid \begin{array}{ccc} X_1^\bullet & \xrightarrow{\alpha} & X_2^\bullet \\ {\scriptstyle f_1^\bullet}\downarrow & & \downarrow{\scriptstyle f_2^\bullet} \\ Y_1^\bullet & \xrightarrow{\beta} & Y_2^\bullet \end{array} \text{ commutes} \right\}$

Definition 4.24. The mapping cone is a functor

$$\begin{array}{rrcl} C^\bullet : & \mathbf{Mor}\left(\mathbf{Com}\left(\mathcal{A}\right)\right) & \longrightarrow & \mathbf{Com}\left(\mathcal{A}\right) \\ & \left(f^\bullet : X^\bullet \to Y^\bullet\right) & \longmapsto & C^\bullet\left(f^\bullet\right) = \left(Y^\bullet \oplus X^\bullet\left[1\right], d_C^\bullet\right) \end{array}$$

where

$$C^\bullet\left(f^\bullet\right) = \left[\cdots \to \begin{array}{c} Y^n \\ \oplus \\ X^{n+1} \end{array} \xrightarrow{\begin{bmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{bmatrix}} \begin{array}{c} Y^{n+1} \\ \oplus \\ X^{n+2} \end{array} \to \cdots \right]$$

Note $d_C^{n+1} \circ d_C^n = \begin{bmatrix} d_Y^n & f^{n+2} \\ 0 & -d_X^{n+2} \end{bmatrix} \begin{bmatrix} d_Y^n & f^{n+1} \\ 0 & -d_X^{n+1} \end{bmatrix} = \begin{bmatrix} d_Y^{n+1} d_Y^n & d_Y^{n+1} f^{n+1} - f^{n+2} d_X^{n+1} \\ 0 & d_X^{n+2} d_X^{n+1} \end{bmatrix} = 0.$
Therefore $C^\bullet(f^\bullet)$ is a well-defined complex. On morphisms, $C^\bullet(\alpha^\bullet, \beta^\bullet) = \beta^\bullet \oplus \alpha^\bullet[1]$.

Example 4.18. Let $f : X \to Y$ be a morphism in $\mathcal{A}$, thinking of f as the morphism between two 0-complexes in $\mathbf{Com}(\mathcal{A})$.

$$C^\bullet(f) = Y \oplus X[1] = \left[0 \to \underset{-1}{X} \xrightarrow{f} \underset{0}{Y} \to 0\right]$$

then

$$H^n(C^\bullet(f)) = \begin{cases} \mathrm{Ker}(f) & n = -1 \\ \mathrm{Coker}(f) & n = 0 \\ 0 & o.w. \end{cases}$$

Remark 4.12. $C^\bullet(f)$ is the replacement of kernel and cokernel in $\mathcal{D}(\mathcal{A}) = \mathbf{Com}(\mathcal{A})\left[W^{-1}\right]$ which is additive but not abelian.

Proposition 4.3. *For $f^\bullet : X^\bullet \to Y^\bullet$, we have f is a quasi-isomorphism if and only if $C^\bullet(f^\bullet)$ is acyclic.*

Proof. There is a natural short exact sequence of complexes for any $f^\bullet : X^\bullet \to Y^\bullet$,

$$0 \longrightarrow Y^\bullet \hookrightarrow C^\bullet(f^\bullet) = Y^\bullet \oplus X^\bullet[1] \twoheadrightarrow X^\bullet[1] \longrightarrow 0$$

i.e.

$$0 \longrightarrow Y^n \hookrightarrow Y^n \oplus X^{n+1} \twoheadrightarrow X^{n+1} \longrightarrow 0$$

is exact for any $n \in \mathbb{Z}$. To see that it's well-defined, we can check

$$\begin{tikzcd}
0 \arrow[r] & Y^n \arrow[r, hook, "i"] \arrow[d, "d_Y^n"'] & Y^n \oplus X^{n+1} \arrow[r, two heads, "p"] \arrow[d, "d_C^n"'] & X^{n+1} \arrow[r] \arrow[d, "d_X^{n+1}"'] & 0 \\
0 \arrow[r] & Y^{n+1} \arrow[r, hook, "i"] & Y^{n+1} \oplus X^{n+2} \arrow[r, two heads, "p"] & X^{n+2} \arrow[r] & 0
\end{tikzcd}$$

commutes by direct computation. Associated long exact sequence in cohomology

$$\cdots \to H^n(Y) \to H^n(C(f)) \to H^n(X[1]) = H^{n+1}(X) \xrightarrow{\delta^n} H^{n+1}(Y) \to H^{n+1}(C(f)) \to H^{n+1}(X) \to \cdots$$

where $\delta^n = H^{n+1}(f)$. Hence f is a quasi-isomorphism if and only if δ^n are isomorphisms for any $n \in \mathbb{Z}$, if and only if $H^n(C(f)) = 0, \forall n \in \mathbb{Z}$, i.e. $C^\bullet(f^\bullet)$ is acyclic. □

Exercise 4.12. Let $\mathbf{Com}^b(\mathcal{A}) \subseteq \mathbf{Com}(\mathcal{A})$ be the full subcategory of bounded complexes, i.e. $X^n = 0, \forall n \gg 0$. Then $\mathbf{Com}^b(\mathcal{A})$ is generated by $\mathcal{A}$ in the sense that any complex can be obtained by taking iterated suspensions and mapping cones. This implies that $\mathcal{D}^b(\mathcal{A})$ is generated (as a triangulated category) by $\mathcal{A}$.

Remark 4.13. Let $\mathcal{C}$ be a pointed model category. $\mathbf{Mor}(\mathcal{C})$ the category of morphisms in $\mathcal{C}$ is a model category.

$$\begin{array}{rccccl} F: & \mathbf{Mor}(\mathcal{C}) & \rightleftharpoons & \mathcal{C} & : G \\ & (* \to X) & \leftarrow\!\shortmid & X & \\ & \left(X \xrightarrow{f} Y\right) & \mapsto & * \sqcup Y = \operatorname{colim}\left\{* \to X \xrightarrow{f} Y\right\} & \end{array}$$

with

$$\mathbb{L}F : \mathrm{Ho}(\mathbf{Mor}(\mathcal{C})) \rightleftharpoons \mathrm{Ho}(\mathcal{C}) : \mathbb{R}G$$

where

$$\mathbb{L}F\left(X \xrightarrow{f} Y\right) = \operatorname{hocolim}\left(* \hookrightarrow X \xrightarrow{f} Y\right) := \operatorname{hocofibr}\left(X \xrightarrow{f} Y\right)$$

If $\mathcal{C} = (\mathbf{Com}(\mathcal{A}), 0)$, then $\mathbb{L}F = \mathcal{C}$. If $\mathcal{C} = \mathbf{Top}_*$, then $\mathbb{L}F$ is the classical mapping cone.

Mapping Cylinder

Given $f^\bullet : X^\bullet \to Y^\bullet$, define $\mathrm{Cyl}(f) = C^\bullet\left(C(f)[-1] \xrightarrow{p[-1]} X\right)$. We can write this explicitly. $\mathrm{Cyl}(f) = Y^\bullet \oplus X^\bullet \oplus X^\bullet[1]$ as graded objects. Differential is given by

$$d^n_{\mathrm{Cyl}} : \begin{array}{c} Y^n \\ \oplus \\ X^n \\ \oplus \\ X^{n+1} \end{array} \xrightarrow{\begin{bmatrix} d^n_Y & 0 & -f^{n+1} \\ 0 & d^n_X & \mathrm{Id}_X \\ 0 & 0 & -d^{n+1}_X \end{bmatrix}} \begin{array}{c} Y^{n+1} \\ \oplus \\ X^{n+1} \\ \oplus \\ X^{n+2} \end{array}$$

Check that $d^{n+1}_{\mathrm{Cyl}} \circ d^n_{\mathrm{Cyl}} = 0, \forall n \in \mathbb{Z}$, so $\mathrm{Cyl}(f)$ is a complex.

There are two natural exact sequences for any $f : X \to Y$.

$$\begin{array}{cccccccc} 0 \to & X^\bullet & \overset{i_X}{\hookrightarrow} & \mathrm{Cyl}(f) & \xrightarrow{\pi_f} & C(f) & \to 0 \\ & x^n & \mapsto & \begin{bmatrix} 0 \\ x^n \\ 0 \end{bmatrix}, \begin{bmatrix} y^n \\ x^n \\ x^{n+1} \end{bmatrix} & \mapsto & \begin{bmatrix} y^n \\ x^{n+1} \end{bmatrix} & \end{array}$$

$$\begin{array}{cccccccc} 0 \to & Y^\bullet & \xrightarrow{\alpha_Y} & \mathrm{Cyl}(f) & \to & C(-\mathrm{Id}_X) & \to 0 \\ & y^n & \mapsto & \begin{bmatrix} y^n \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} y^n \\ x^n \\ x^{n+1} \end{bmatrix} & \mapsto & \begin{bmatrix} x^n \\ x^{n+1} \end{bmatrix} & \end{array}$$

Proposition 4.4. *We have*

1. *$f^\bullet$ is a quasi-isomorphism if and only if i_X is a quasi-isomorphism.*

2. *For any $f^\bullet : X^\bullet \to Y^\bullet$, α_Y is a quasi-isomorphism.*

Proof. $f^\bullet$ is a quasi-isomorphism if and only if $H^n(C(f)) = 0, \forall n \in \mathbb{Z}$, if and only if every $H^n(i_X)$ is isomorphism.

Id_X is a quasi-isomorphism, so $C(-\mathrm{Id}_X)$ is acyclic, hence α_Y is a quasi-isomorphism. □

Exercise 4.13. Show that the morphism

$$\begin{array}{rccc} \beta^\bullet : & \mathrm{Cyl}\,(f) & \to & Y^\bullet \\ & \begin{bmatrix} y^n \\ x^n \\ x^{n+1} \end{bmatrix} & \mapsto & y^n + f^n\,(x^n) \end{array}$$

is well-defined and $\beta \circ \alpha = \mathrm{Id}_Y$ and $\alpha \circ \beta \simeq \mathrm{Id}_{\mathrm{Cyl}(f)}$, thus α is a homotopy equivalence.

Proof. By direct computation, $f\beta^n \circ \alpha^n\,(y^n) = y^n$, so $\beta \circ \alpha = \mathrm{Id}_Y$. And

$$\alpha^n \circ \beta^n \left(\begin{bmatrix} y^n \\ x^n \\ x^{n+1} \end{bmatrix} \right) = \begin{bmatrix} y^n + f^n\,(x^n) \\ 0 \\ 0 \end{bmatrix},$$

check that

$$s : \begin{bmatrix} y^n \\ x^n \\ x^{n+1} \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ x^n \end{bmatrix}$$

defines a chain homotopy between $\alpha \circ \beta$ and $\mathrm{Id}_{\mathrm{Cyl}(f)}$. □

4.9 Classical Derived Functor

Philosophically (classical) interesting objects in abelian categories are expressed in terms of exact sequences of simple objects, e.g. $0 \to X_1 \to X \to X_2 \to 0$. What happens if we apply additive functors to interesting objects? Unfortunately, interesting functors are usually not exact. If $0 \to FX_1 \to FX \to FX_2 \to 0$ is not exact, we cannot express FX in terms of FX_1 and FX_2.

Idea: we associate to F some correction in a universal way, the derived functors, to restore correctness.

δ-functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories.

Definition 4.25. A left *δ-functor* is a sequence $T = (T_i, \delta_i)_{i\geq 0}$ of additive functors $T_i : \mathcal{A} \to \mathcal{B}$ and morphisms of functors $\delta_i : T_{i+1}(X_2) \to T_i(X_1)$ on the category $\mathbf{Exc}(\mathcal{A})$ of short exact sequences.

$$\begin{tikzcd} (0 \to X_1 \to X \to X_2 \to 0) \arrow[r, mapsto] & T_{i+1}(X_2) \arrow[d] \\ (0 \to X_1 \to X \to X_2 \to 0) \arrow[r, mapsto] & T_i(X_1) \end{tikzcd}$$

such that for any short exact sequence $0 \to X_1 \to X \to X_2 \to 0$ in $\mathbf{Exc}(\mathcal{A})$, we have a long exact sequence

$$\cdots \to T_{i+1}(X_1) \to T_{i+1}(X) \to T_{i+1}(X_2) \xrightarrow{\delta_i} T_i(X_1) \to T_i(X) \to T_i(X_2) \xrightarrow{\delta_{i-1}} T_{i-1}(X_1) \to \cdots \to T_0(X_1) \to T_0(X$$

Definition 4.26. A left δ-functor is *universal* if for any left δ-functor $T' = (T'_i, \delta'_i)_{i\geq 0}$ given together with $f_0 : T_0 \to T'_0$, there exists a unique extension $f_i : T_i \to T'_i$ commuting with δ_i's.

Definition 4.27. (Grothendieck) Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor, then its *classical left derived functor* $\mathbb{L}F$ is a universal left δ-functor $(\mathbb{L}_i F, \delta_i)_{i\geq 0}$ with the property $\mathbb{L}_0 F \cong F$.

Dually, we can define right δ-functors $T = (T_i, \delta_i)_{i\geq 0}$ of additive functors $T_i : \mathcal{A} \to \mathcal{B}$ and morphisms of functors $\delta_i : T_i(X_1) \to T_{i+1}(X_2)$ such that for any short exact sequence $0 \to X_1 \to X \to X_2 \to 0$ in $\mathbf{Exc}(\mathcal{A})$, we have a long exact sequence

$$0 \to T_0(X_2) \to T_0(X) \to T_0(X_1) \xrightarrow{\delta_0} T_1(X_2) \to \cdots \to T_{i-1}(X_1) \xrightarrow{\delta_{i-1}} T_i(X_2) \to T_i(X) \to T_i(X_1) \xrightarrow{\delta_i} T_{i+1}(X_2) -$$

Definition 4.28. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor, then its *classical right derived functor* $\mathbb{R}F$ is a universal right δ-functor $(\mathbb{R}_i F, \delta_i)_{i\geq 0}$ with the property $\mathbb{R}_0 F \simeq F$.

Main Problem: which conditions on $\mathcal{A}$ and $\mathcal{B}$ and F ensure the existence of derived functors? How to compute them?

There are different conditions, but the simplest and universal (in the sense that this condition only depends on $\mathcal{A}$) condition is given by the following theorem.

Theorem 4.8. *If $\mathcal{A}$ has enough projectives, then every right exact functor has a left derived functor.*

Theorem 4.9. *If $\mathcal{A}$ has enough injectives, then every left exact functor has a right derived functor.*

Construction: Assume that $\mathcal{A}$ has enough projectives. Given any object $X \in Ob(\mathcal{A})$, there is an epimorphism $\varepsilon^0 : P^0 \to X$ with P^0 projective in $\mathcal{A}$.

Take $X^{-1} = \operatorname{Ker}(\varepsilon^0) \in Ob(\mathcal{A})$, there exists $\varepsilon^1 : P^{-1} \to X^{-1}$ with P^{-1} projective. Iteratively we can get a complex

$$P^\bullet = \left[\cdots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \to 0\right]$$

in $\mathcal{A}$ with each $P^i, i \le 0$ projective and $\varepsilon^\bullet : P^\bullet \to X$ is a surjective quasi-isomorphism, where $X = [0 \to X \to 0]$ as a complex.

Definition 4.29. If $\mathcal{A}$ has enough projective, every $X \in Ob(\mathcal{A})$ can be covered by quasi-isomorphism $\varepsilon^\bullet : P^\bullet \to X^\bullet$, and it's called a *projective resolution* of X.

Exercise 4.14. Let $X^\bullet$ be a bounded complex ($X^i = 0, i \gg 0$) in $\mathbf{Com}(\mathcal{A})$, then there exists $P^\bullet$ with quasi-isomorphism $\varepsilon^\bullet : P^\bullet \to X^\bullet$ called a *projective resolution* of $X^\bullet$.

Projective resolutions are not unique, but they have the following properties.

Lemma 4.11. *If $f : X \to Y$ is a morphism in $\mathcal{A}$, $\varepsilon_X : P^\bullet \to X$ and $\varepsilon_Y : Q^\bullet \to Y$ are two projective resolutions, then there exists $\tilde{f}^\bullet : P^\bullet \to Q^\bullet$ in $\mathbf{Com}(\mathcal{A})$ such that*

$$\begin{tikzcd}
P^\bullet \arrow[r, "\tilde{f}^\bullet"] \arrow[d, two heads, "\varepsilon_X"'] & Q^\bullet \arrow[d, two heads, "\varepsilon_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^\bullet(P^\bullet) \arrow[r, "H^\bullet(\tilde{f}^\bullet)"] \arrow[d, "\varepsilon_X"', "\wr"] & H^\bullet(Q^\bullet) \arrow[d, "\varepsilon_Y"', "\wr"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where $H^\bullet\left(\tilde{f}^\bullet\right) \cong f$.

Proof. Put $P' = X, d_P^0 = \varepsilon_X$, $Q' = Y, d_Q^0 = \varepsilon_Y$, then

$$\left[\cdots \to P^{-2} \xrightarrow{d^{-2}} P^{-1} \to P^0 \xrightarrow{d_P^0} P' \to 0\right]$$

and

$$\left[\cdots \to Q^{-2} \xrightarrow{d^{-2}} Q^{-1} \to Q^{0} \xrightarrow{d_Q^0} Q' \to 0\right]$$

are acyclic, with $P_i, Q_j, i, j \leq 0$ projective.

Put $f' = f : P' \to Q'$, we will prove the existence of $f^k : P^k \to Q^k$ by induction on k. Assume that $f^{k+1}, f^{k+2}, \cdots, f^0$ exists, define $\varphi^k = f^{k+1} \circ d_P^k$. Since $d_Q^{k+1} \circ \varphi_k = f^{k+2} \circ d_P^{k+1} \circ d_Q^k = 0$, $\mathrm{Im}\varphi \subseteq \mathrm{Ker}\left(d_Q^{k+1}\right) = \mathrm{Im}\left(d_Q^k\right)$. By projectivity of P^k, there exists $f^k : P^k \to Q^k$such that $d_Q^k \circ f^k = \varphi^k = f^{k+1} \circ d_P^k$.

$$\begin{tikzcd}
P^k \arrow[r, "d_P^k"] \arrow[d, dashed, "f^k"'] \arrow[dr, "\varphi^k"] & P^{k+1} \arrow[r, "d_P^{k+1}"] \arrow[d, "f^{k+1}"] & P^{k+2} \arrow[d, "f^{k+2}"] \\
Q^k \arrow[r, "d_Q^k"] & Q^{k+1} \arrow[r, "d_Q^{k+1}"] & Q^{k+2}
\end{tikzcd}$$

Next, if there are two lifts $f^\bullet, g^\bullet : P^\bullet \to Q^\bullet$ are two lifts of $X \xrightarrow{f} Y$, we need to show there exists $h^k : P^k \to Q^{k-1}$such that $f^k - g^k = h^{k+1} \circ d_P^k + d_Q^{k-1} \circ h^k$.

$$\begin{tikzcd}
\cdots \arrow[r] & P^{k-1} \arrow[r, "d_P^{k-1}"] \arrow[d, shift right, "f^{k-1}"'] \arrow[d, shift left, "g_{k-1}"] & P^k \arrow[r, "d_P^k"] \arrow[dl, "h^k"] \arrow[d, shift right, "f^k"'] \arrow[d, shift left, "g_k"] & P^{k+1} \arrow[r] \arrow[dl, "h^{k+1}"'] \arrow[d, shift right, "f^{k+1}"'] \arrow[d, shift left, "g_{k+1}"] & \cdots \\
\cdots \arrow[r] & Q^{k-1} \arrow[r, "d_Q^{k-1}"] & Q^k \arrow[r, "d_Q^k"] & Q^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Set $h^0 = 0$ and construct h^k by induction. Assume there exists $h^{k+1}, h^{k+2}, \cdots, h^0$, define $\psi^k = f^k - g^k - h^{k+1} \circ d_P^k : P^k \to Q^k$ and notice that $d_Q^k \circ \psi^k = \left(f^{k+1} - g^{k+1} - d_Q^k \circ h^{k+1}\right) \circ d_P^k = h^{k+2} \circ d_P^{k+1} \circ d_P^k = 0$, so $\mathrm{Im}\left(\psi^k\right) \subseteq \mathrm{Ker}\left(d_Q^k\right) = \mathrm{Im}\left(d_Q^{k-1}\right)$, therefore there exists $h^k : P^k \to Q^k$ such that $d_Q^{k-1} \circ h^k = \psi^k = f^k - g^k - h^{k+1} \circ d_P^k$. □

Corollary 4.10.

1. *If $\varepsilon_X : P^\bullet \to X$ and $\tilde{\varepsilon}_X : \tilde{P}^\bullet \to X$ are two resolutions, then there exists $f^\bullet : P^\bullet \to \tilde{P}^\bullet$ and $g^\bullet : \tilde{P}^\bullet \to P^\bullet$ such that*

$$\begin{tikzcd}
P^\bullet \arrow[r, shift left, "f^\bullet"] \arrow[d] & \tilde{P}^\bullet \arrow[l, shift left, "g^\bullet"] \arrow[d] \\
X \arrow[r, "Id_X", "="'] & X
\end{tikzcd}$$

and $f^\bullet \circ g^\bullet \simeq Id^\bullet_{\tilde{P}}$, $g^\bullet \circ f^\bullet \simeq Id^\bullet_P$.

2. *If* $f^\bullet : P^\bullet \to Q^\bullet$ *is a morphism from projective to acyclic complexes, then* $f^\bullet \simeq 0$.

Construction of Left Derived Functors

Assume that $\mathcal{A}$ has enough projections. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. To define $\mathbb{L}F$, choose for any $X \in Ob(\mathcal{A})$ a projective resolution $\varepsilon_X : P^\bullet \to X$, define $\mathbb{L}_i F : \mathcal{A} \to \mathcal{B}$ on

Objects: $(\mathbb{L}_i F)(X) = H^{-i}(F(P^\bullet))$

Morphisms: for $f : X \to Y$, define a lift $\tilde{f} : P^\bullet_X \to P^\bullet_Y$, and $(\mathbb{L}_i F)(f) = H^{-i}\left(F\left(\tilde{f}\right)\right)$

Lemma 4.11 implies that this definition is well-defined, i.e. independent of the choice of $\tilde{f}$ and the choice of resolution.

Theorem 4.10. *(Constructive)* $(\mathbb{L}_i F)_{i\geq 0}$ *is the classical left derived functor of* F.

Proof. Step 1: Construct δ_i making $(\mathbb{L}_i F, \delta_i)_{i\geq 0}$ a left δ-functor.

Step 2: Show that $\mathbb{L}_0 F \cong F$. Apply F to the resolution $P^{-1} \to P^0 \to X \to 0$, $X \in Ob(\mathcal{A})$. Since F is right exact, $FP^{-1} \xrightarrow{Fd^{-1}} FP^0 \to FX \to 0$ is exact, so $H^0\left(FP^0\right) = \operatorname{Coker}\left(Fd^{-1}\right) \cong F\left(\operatorname{Coker} d^{-1}\right) \cong FX$.

Step 3: Show universality. Use the following Grothendieck's lemma. Note that $\mathbb{L}_i F(P) = 0, \forall i \geq 0$ whenever P is projective, because if P is projective, $[0 \to P \to 0] \xrightarrow{\mathrm{Id}} P$ is a projective resolution of P, then $FP = [0 \to P \to 0]$, and $H^i(FP) = 0, \forall i \neq 0$. Therefore $(\mathbb{L}_i F, \delta_i)_{i\geq 0}$ is coeffaceable, and hence universal. □

Definition 4.30. A functor $F : \mathcal{A} \to \mathcal{B}$ is called *coeffaceable* if for any $X \in Ob(\mathcal{A})$, there exists $P \in Ob(\mathcal{A})$ with epimorphism $f : P \twoheadrightarrow X$ such that $F(f) = 0$.

Lemma 4.12. *If* $T = (T_i, \delta_i)_{i\geq 0}$ *is a left* δ*-functor such that* T_i *is coeffaceable, then* T *is universal.*

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